OPEN ACCESS

International Journal Of Modern Engineering Research (IJMER)

Common Fixed Point Theorem for Three Pair of Mappings in Semi-Metric Space Using ∅-Contraction

[1*]U. Rajopadhyayaand[2]K.Jha

1.Department of Management Sciences (Mathematics), School of Management, Kathmandu University, Nepal
2. Department of Mathematics, Kathmandu University, Nepal.
**Corresponding Author: U. Rajopadhyayaand*

Abstract: *In this paper, we establish a common fixed point theorem for three pairs of self mappings in semi-metric space which improves and extends similar known results in the literature.*
Keywords: *Semi-metric space, Compatibility, Weakly compatible, fixed point and common fixed point.*
AMS Subject Classification : *Primary 54H25, Secondary 47H10.*

Date of Submission: 07-08-2020 Date of acceptance: 21-08-2020

I. INTRODUCTION:

Fixed point theory in semi-metric space is one of the emerging area of interdisciplinary mathematical research. It plays a crucial role in nonlinear functional analysis. S.Banach published his contraction principle published in 1922.Since then this principle has been extended and generalized in several ways in semi-metric space. Among which one such generalization is formulated in semi-metric space initiated by M.Frechet [6], K.Menger [12] and W.A.Wilson[14]. In 1976 M. Cicchese [5] introduced the notion of a contractive mapping in semi-metric space and proved the first fixed point theorem for this class of spaces. Further fixed point results for the class of spaces were obtained by T.L.Hicks, B.E.Rhoades [7], M.Aamri and D.ElMoutawakil [1], M.Imdad, J. Ali and L.Khan[9]. G. Jungck[11] introduced the concept of weakly compatible mapping. This concept has been frequently used to prove existence theorem in common fixed point theory. However, the study of common fixed point theorem for non-compatible mapping also has become interesting notion. S.H.Cho, G.Y.Lee and J.S.Bae [4] initially proved some common fixed point theorems for non-compatible mappings and gave a notion E.A. which generalizes the concept of non-compatible mappings in metric space and also proved some common fixed point theorems for strict contractive noncompatible mappings in metric space.

In this paper, we prove a common fixed point theorem for sixself-mappings using weakly compatible and E.A. property that extends the results of M.Aamri and D.El.Moutawakil[1] and other similar results.

II. DEFINITION

Definition 2.1[9]Let X be a non-empty set and $d: X \times X \to [0, \infty)$. (X, d) be a **semi-metric space** (symmetric space) if and only if it satisfies the following:

W1 :$d(x, y) = 0$ if and only if $x = y$.

W2 :$d(x, y) = d(y, x)$ if and only if $x = y$ for any $x, y \in X$.

The difference of a semi-metric and a metric comes from the triangle inequality. In order to obtain fixed point theorems on a semi-metric space, we need some additional axioms as a partial replacement for triangle inequality which are as follows.

Let (X, d) be a semi-metric space. Then, for sequence $\{x_n\}, \{y_n\}$ in X and $x \in X$, we have

W4[14]: $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x_n) = 0$ imply $\lim_{n\to\infty} d(y_n, x) = 0$.

H.E[1].: $\lim_{n\to\infty} d(x_n, x) = 0$ $\lim_{n\to\infty} d(y_n, x) = 0$ imply $\lim_{n\to\infty} d(y_n, x_n) = 0$.

Definition 1.1 [1] Let A and B be two self-mappings of a semi-metric space (X, d). Then, A and B are said to be compatible if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$, for some $t \in X$.

Definition 2.2.[1] Let A and B be two self-mappings of a semi-metric space (X, d).Then, A and B satisfy the property E.A. if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$, for some $t \in X$.

Definition 2.3.[1] Let A and B be two self-mappings of a semi-metric space (X, d). Then, A and B are said to be weakly compatible if they commute at their coincidence points.

In order to establish our result, we need a function $\emptyset: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $0 < \emptyset(t) < t,\ t > 0$.

III. MAIN RESULTS:

Theorem 3.1: Let (X,d) be a semi-metric space that satisfies (W4) and (HE) . Let A,B,T,S,P and Q be self-mappings of X such that

i) $ABX \subset PX$ and $TSX \subset QX$

ii) $d(ABx, TSy) \le \emptyset(\max\{d(Qx, Py), d(Qx, TSy), d(Py, TSy)\})$ for all $(x, y) \in X \times X$

iii) (AB, Q) or (TS, P) satisfies the property E.A. , and

iv) (AB, Q) and (TS, P) are weakly compatibles.

If the range of the one of the mapping AB, TS,P and Q is a complete subspace of X then AB, TS,P and Q have a unique common fixed point. Furthermore if the pairs (A, B), (A, P), (B, P),(S,T), (S, J) and (T, Q) are commuting pair of mappings then A,B,T,S,P and Q have a unique common fixed point.

Proof:

Suppose that (TS, P) satisfies the property E.A. , then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(TSx_n, t) = \lim_{n\to\infty} d(Px_n, t) = 0$ for some $t \in X$. Hence by the property (HE),we get $\lim_{n\to\infty} d(TSx_n, Px_n) = 0$. Since $TSX \subset QX$ there exists a sequence $\{y_n\}$ in X such that $TSx_n = Qy_n$. Hence,we get, $\lim_{n\to\infty} d(Qy_n, t) = 0$.

We prove that $\lim_{n\to\infty} d(ABy_n, t) = 0$. Using condition (i)

$$\begin{aligned} d(ABy_n, TSx_n) &\le \emptyset(\max\{d(Qy_n, Px_n), d(Qy_n, TSx_n), d(Px_n, TSx_n)\}) \\ &= \emptyset(\max\{d(TSx_n, Px_n), d(Qy_n, Qy_n), d(Px_n, TSx_n)\}) \\ &= \emptyset(\max\{d((TSx_n, Px_n), 0, d(Px_n, TSx_n)\}) \\ &= \emptyset\ (d(TSx_n, Px_n)) \\ &< d(TSx_n, Px_n). \end{aligned}$$

Letting $\to \infty$, we have , $\lim_{n\to\infty} d(ABy_n, TSx_n) = 0$. By the property (W4), we have $\lim_{n\to\infty} d(ABy_n, t) = 0$.

Suppose QX is a complete subspace of X. Then $Qu = t$ for some $u \in X$,

Also, we have

$$\lim_{n\to\infty} d(ABy_n, Qu) = \lim_{n\to\infty} d(TSx_n, Qu) = \lim_{n\to\infty} d(Px_n, Qu) = \lim_{n\to\infty} d(Qy_n, Qu) = 0$$

Using condition (ii) it follows that

$d(ABu, TSx_n) \le \emptyset(\max\{d(Qu, Px_n), d(Qu, TSx_n), d(Px_n, TSx_n)\})$.

Letting $n \to \infty$, we have ,

$\lim_{n\to\infty} d(ABu, TSx_n) = 0$, By the property (W4), we have $\lim_{n\to\infty} d(ABy_n, ABu) = 0$. This implies $\lim_{n\to\infty} d(t, ABu) = 0$ and hence $\lim_{n\to\infty} d(Qu, ABu) = 0$. Therefore, we have $Qu = ABu$.

The weak compatibility of AB and Q implies that $ABQu = QABu$, then we have

$ABABu = ABQu = QABu = QQu$.

Again, since $ABX \subset PX$, so there exists $z \in X$, such that $ABu = Pz$. This implies $ABu = Qu = Pz$. We claim that $Pz = TSz$. If not, condition (ii) gives,

$$\begin{aligned} d(ABu, TSz) &\le \emptyset(\max\{d(Qu, Pz), d(Qu, TSz), d(Pz, TSz)\}) \\ &= \emptyset(\max\{d(ABu, ABu), d(ABu, TSz), d(ABu, TSz)\} \\ &= \emptyset\ (d(ABu, TSz)) \\ &< d(ABu, TSz), \end{aligned}$$

Which is contradiction. Therefore, we get $Pz = TSz$. Hence, we get $ABu = Qu = Pz = TSz$.

The weak compatibility of TS and P imply that $TSPz = PTSz$ and

$$PPz = PTSz = TSPz = TSTSz.$$

Now, We prove that ABu is a common fixed point of AB, TS,P and Q. Suppose that $AB(ABu) \ne ABu$. Then, we have,

$$\begin{aligned} d\big(ABu, AB(ABu)\big) &= d(AB(ABu), TSz) \\ &\le \emptyset(\max\{d(QABu, Pz), d(QABu, TSz), d(Pz, TSz)\}) \\ &= \emptyset(\max\{d(ABABu, ABu), d(ABABu, ABu), d(Pz, Pz)\}) \\ &= \emptyset(\ d(ABABu, ABu)) \\ &< d(ABABu, ABu), \end{aligned}$$

Which is contradiction. Therefore, we get $ABu = AB(ABu) = Q(ABu)$.
Hence, ABu is a common fixed point of AB and Q. Similarly, we can prove that TSz is a common fixed point of TS and P. Since $ABu = TSz$, we conclude that ABu is a common fixed point of AB, TS, P and Q.
The proof is similar when PX is assumed to be a complete subspace of X. The case in which ABX or TSX is a complete subspace of X are similar to the case in which PX or QX respectively is complete since $ABX \subset PX$ and $TSX \subset QX$.
Since ABu is a common fixed point of AB, TS, P and Q. We can write
$AB(ABu) = TS(ABu) = P(ABu) = Q(ABu) = ABu$.
If v is another common fixed point of AB, TS, P and Q, then for $p \in X$ and $v \neq ABu$, we can write
$AB(v) = TS(v) = P(v) = Q(v) = v$.
Therefore, we have

$$
\begin{aligned}
d(ABu, v) &= d(AB(ABu), TSv) \\
&\leq \emptyset(\max\{(d(Q(ABu), Pv), d(Q(ABu), TSv), d(Pv, TSv)\}) \\
&= \emptyset(\max\{d(Q(ABu), Pv), d(Q(ABu), Pv), d(Pv, Pv)\}) \\
&= \emptyset(d(Q(ABu), Pv)) \\
&= \emptyset(d(ABu, v)) \\
&< d(ABu, v),
\end{aligned}
$$

Which is contradiction. Therefore, we have $ABu = v$, Hence AB, TS, P and Q have unique common fixed point.
We need to show that v is only the common fixed point of the family $F = \{A, B, T, S, P, Q\}$ when the pairs (A, B), (A, P), (B, P), (S, T), (S, Q) and (T, Q) are commuting mappings. For this, we can write,
$Av = A(ABv) = A(BAv) = AB(Av)$,
$Av = A(Pv) = P(Av)$,
$Bv = B(ABv) = BA(Bv) = AB(Bv)$, and
$Bv = B(Pv) = P(Bv)$.
This shows that Av and Bv are common fixed point of (AB, P). This implies that
$Av = v = Bv = Pv = ABp$. Similarly, we have $Tv = v = Sv = Qv = TSv$.
Thus, A, B, T, S, P and Q have a unique common fixed point T.
This completes the proof.

Example 3.1. Consider $X = [0, 1]$ with the semi-metric space (X, d) defined by $d(x, y) = (x - y)^2$. Define a self map A, B, T, S, P and Q as $Ax = \frac{3x}{4}$, $Bx = \frac{4x}{5}$, $S(x) = \frac{2x}{5}$, $Tx = \frac{5x}{6}$, $P(x) = \frac{2x}{3}$ and $Q(x) = \frac{9x}{10}$. Then d satisfies W4 and H.E. For the sequence $x_n = \frac{1}{n}$. The mappings satisfy all the conditions of above theorem 2.1 and hence they have a unique common fixed point $x = 0$.
It is noted that the above theorem holds true if condition (iii) is replaced by (AB, Q) and (TS, P) satisfies the property E.A.
In the above Theorem 2.1.if we take $A = B$ and $T = S$, then we have the following corollary.

Corollary 3.1. : Let (X,d) be a semi-metric space that satisfies (W4) and (HE) . Let A, T, P and Q be self-mappings of X such that
i) $AX \subset PX$ and $TX \subset QX$
ii) $d(Ax, Ty) \leq \emptyset(\max\{d(Qx, Py), d(Qx, Ty), d(Py, Ty)\})$ for all $x, y \in X \times X$
iii) (A, Q) or (T, P) satisfies the property E.A. , and
iv) (A, Q) and (T, P) are weakly compatibles.
If the range of the one of the mapping A, T, P and Q is a complete subspace of X then A, T, P and Q have a unique common fixed point.
This is the result of M. Aamri and D. El. Moutawakil [1].
In theorem 2.1, if we take $A = B = P$ and $T = S = Q$ we have the following corollary.

Corollary 3.2 : Let (X,d) be a semi-metric space that satisfies (W4) and (HE) . Let A and T be self- mappings of X such that
i) $AX \subset TX$
ii) $d(Ax, Ty) \leq \emptyset(\max\{d(Tx, Ay), d(Tx, Ty), d(Ay, Ty)\})$ for all $x, y \in X \times X$
iii) A and T satisfy the property E.A. , and
iv) A and T are weakly compatibles.
If the range of the one of the mapping A and T is a complete subspace of X then A, and T have a unique common fixed point.
This is the result of M. Aamri and D. El. Moutawakil [1].

IV. CONCLUSION

Our result generalizes the result of M.Aamri and D.El. Moutawakil [1], extends the results ofT.L.Hicks and B.E.Rhoades [7],M. Imdad and Q. H. Khan [8] and other similar results in semi-metric space .

REFERENCES

[1]. **M. Aamri and D. El. Moutawakil,***Common fixed points under contractive conditions in symmetric space*, Applied Mathematics E-Notes, **3**(2003), 156-162.
[2]. **I.D.Arandelovic and D.S. Petkovic,***A note on some fixed point results*, Applied Mathematics E-Notes, **9**(2009), 254-261.
[3]. **E.W.Chittenden**, *On the equivalence of ecartandvoisinage*,Trans.Amer.Math.Soc,18(1917), 1661-166.
[4]. **S.H.Cho, G.Y.Lee** and**J.S.Bae**, *On concidence and fixed point theorems in symmetric space*,Fixed Point Theory Appl. **2008**(2008), Art.ID 562130, 9 pp.
[5]. **M. Cicchese**, *Questioni di comptetezza e contrazioni in spazimetrici generalization* Boll.Un.Mat.Ital.,13-A(5)(1976), 175-179.
[6]. **M. Fr´echet**, *Sur quelques points du calculfonctionnel*, Rend. Circ. Mat. Palermo**22** (1906) 1–74.
[7]. **T.L.Hicks**and**B.E.Rhoades**, *Fixed point theory in symmetric space with applications to probabilistic space*, Nonlinear Analysis, 36(1999), 331-334.
[8]. **M. Imdad**and **Q.H.Khan**, *Six mappings satisfying a rational inequality*, RadoviMatematicki, **9**(1999), 251-260.
[9]. **M.Imdad, J. Ali,** and**L.Khan,***Coincidence and fixed points in semi-metric spaces under strict conditions*, J.Math.Anal.Appl.**320**(2006), 352-360.
[10]. **M.Imdad**and **J. Ali,***Common fixed point theorems in semi-metric space employing a new implicit function and common property (E.A.)*, Bull. Belg.Soc.Simon stevin, **16**(2009),421-433.
[11]. **G.Jungck**, *Compatible mappings and common fixed point*, International Journal of Mathematics and Mathematical Sciences, 9(1986),771-779.
[12]. **K. Menger,***Untersuchungenuberallgemeine*, Math Annalen, 100(1928),75-163.
[13]. **R.P. Pant,***Common fixed point of non-commuting mappings*, Journal of mathematical analysis and applications 188(1994), 436-440.
[14]. **W.A.Wilson**,*On semi-metric space*, Amer.J.Math, 53(1931),361-373.

U. Rajopadhyayaand, et. al." Common Fixed Point Theorem for Three Pair of Mappings in Semi-Metric Space Using ∅-Contraction." *International Journal of Modern Engineering Research (IJMER)*, vol. 10(07), 2020, pp 01-04.