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International
Journal
Of Modern Engineering Research (IJMER)

Dynamics Analysis of the Fractional-Order Lagrange System

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Abstract. *The main purpose of this paper is to study the fractional-order model with Caputo derivative associated to Lagrange system. For this fractional-order system we investigate the existence and uniqueness of solutions of initial value problem, asymptotic stability of its equilibrium states, stabilization problem using appropriate controls and numerical integration 1via the fractional Euler method.*

Date of Submission: 11-08-2022 Date of Acceptance: 27-08-2022

I. Introduction

The dynamic behavior of differential systems (in particular, Hamilton-Poisson systems) have been studied due to their deep applications in many areas of science and engineering including robotics, spatial dynamics, complete synchronization and secure comunications. Three remarkable classes of systems formed by differential equations on $\mathbf{R}^3$ are the general Euler top system, the systems of Maxwell-Bloch type and the family of Lotka-Volterra systems. In the recent decades, a series of dynamical systems belonging to these three classes have been studied from the point of view of Poisson geometry by many researchers, for instance [11, 24, 23, 22, 14, 12, 18, 10, 17].

An interesting example of a differential system of the Euler top type is the *Lagrange system* [24]. It is described by the following differential equations on $\mathbf{R}^3$:

$$\dot{x}^1(t) = x^2(t)x^3(t), \quad \dot{x}^2(t) = x^1(t)x^3(t), \quad \dot{x}^3(t) = x^1(t)x^2(t), \tag{1.1}$$

where $\dot{x}^i(t) = dx^i(t)/dt$ and t is the time.

The system (1.1) is used in theoretical physics for the study of $SU(2)-$ monopoles.

The fractional calculus deals with derivations and integration of arbitrary order and has deep and natural connections with many fields of applied mathematics, mathematical physics, engineering, biological systems, control processing, chaos synchronization, [19, 21, 1, 3, 9, 20, 8, 5, 4, 6, 2]. The fractional models associated to nonlinear dynamical systems have been investigated in the papers [16, 13, 15].

This paper is structured as follows. The fractional-order Lagrange system is defined in Section 2. The existence and uniqueness of solutions of initial value problem for the fractional model (2.2) is discussed. For the asymptotic stabilization problem of fractional model (2.2), we associate the fractional Lagrange system with controls c_1 and c_2, denoted by (3.3). In Propositions (3.1) − (3.3) are established sufficient conditions on parameters c_1 and c_2 to control the chaos in the fractional-order Lagrange system

[1] *AMS classification:* 26A33, 53D05, 65P20, 70H05.
Key words and phrases: fractional-order Lagrange system, stability, fractional Euler method, numerical integration.

(3.3). In Section 4, we give the numerical integration of fractional-order Lagrange system (3.3), via the fractional Euler's method. Finally, a numerical simulation of fractional model (3.3) is shown to verify the theoretical results.

2 The fractional-order Lagrange system

We recall the Caputo definition of fractional derivatives, which is often used in concrete applications. Let $f \in C^{\infty}(\mathbf{R})$ and $q \in \mathbf{R}, q > 0$. The $q-$order Caputo differential operator [8], is described by $D_t^q f(t) = I^{m-q} f^{(m)}(t),\ q > 0,$ where $f^{(m)}(t)$ represents the $m-$order derivative of the function f, $m \in \mathbf{N}^*$ is an integer such that $m-1 \leq q \leq m$ and I^q is the $q-$*order Riemann-Liouville integral operator* [21, 8], which is expressed by:

$$I_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) ds,\ q > 0, \quad s \in [0, T], \tag{2.1}$$

where Γ is the Euler Gamma function.
If $q = 1$, then $D_t^q f(t) = df/dt$.

The *fractional-order Lagrange system* associated to dynamics (1.1) is defined by the following set of fractional differential equations:

$$\begin{cases} D_t^q x^1(t) &= x^2(t) x^3(t) \\ D_t^q x^2(t) &= x^1(t) x^3(t), \quad q \in (0,1), \\ D_t^q x^3(t) &= x^1(t) x^2(t). \end{cases} \tag{2.2}$$

The initial value problem of the fractional-order Lagrange system (2.1) can be represented in the following matrix form:

$$D_t^{\alpha} x(t) = x^1(t) A x(t) + x^3(t) B x(t), \qquad x(0) = x_0, \tag{2.3}$$

where $0 < q < 1,\ x(t) = (x^1(t), x^2(t), x^3(t))^T,\ t \in (0, \tau)$ and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proposition 2.1 *The initial value problem of the fractional-order Lagrange system* (2.1) *has a unique solution.*

Proof. Let $f(x(t)) = x^1(t) A x(t) + x^3(t) B x(t)$. It is obviously continuous and bounded on $D = \{x \in \mathbf{R}^3 |\ x^i \in [x_0^i - \delta, x_0^i + \delta]\}, i = \overline{1,3}$ for any $\delta > 0$. We have $f(x(t)) - f(y(t)) = x^1(t) A x(t) - y^1(t) A y(t) + x^3(t) B x(t) - y^3(t) B y(t) = g(t) + h(t)$, where $g(t) = x^1(t) A x(t) - y^1(t) A y(t)$ and $h(t) = x^3(t) B x(t) - y^3(t) B y(t)$. Then

(a) $|f(x(t)) - f(y(t))| \leq |g(t)| + |h(t)|$.

It is easy to see that $g(t) = (x^1(t) - y^1(t)) A x(t) + y^1(t) A (x(t) - y(t))$. Then

$|g(t)| \leq |(x^1(t) - y^1(t)) A x(t)| + |y^1(t) A (x(t) - y(t))| \leq$
$\leq \|A\| (|x(t)| \cdot |x^1(t) - y^1(t)| + |y^1(t)| \cdot |x(t) - y(t)|,$
where $\|\cdot\|$ and $|\cdot|$ denote matrix norm and vector norm respectively.

Using the inequality $|x^1(t) - y^1(t))| \leq |x(t) - y(t))|$ one obtains

$(b)\quad |g(t)| \leq (\|A\| + |y^1(t)|) \cdot |x(t) - y(t)|.$
Similarly, we have
$(c)\quad |h(t)| \leq (\|B\| + |y^3(t)|) \cdot |x(t) - y(t)|.$
According to (b) and (c), the relation (a) becomes
$(d)\quad |f(x(t)) - f(y(t))| \leq (\|A\| + \|B\| + |y^1(t)| + |y^3(t)|) \cdot |x(t) - y(t)|.$
Replacing $\|A\| = 1,\ \|B\| = \sqrt{2}$ and using the inequalities $|y^i(t)| \leq |x_0| + \delta,\ i = 1, 3$ from the relation (d), we deduce that
$(e)\quad |f(x(t)) - f(y(t))| \leq L \cdot |x(t) - y(t)|,\qquad$ where $L = 1 + \sqrt{2} + 2(|x_0| + \delta) > 0.$

The inequality (e) shows that $f(x(t))$ satisfies a Lipschitz condition. Based on the results of Theorems 1 and 2 in [7], we can conclude that the initial value problem of the system (2.2) has a unique solution. □

For the fractional Lagrange system (2.1) we introduce the following notations:

$$f_1(x) = x^2x^3, \quad f_2(x) = x^1x^3, \quad f_3(x) = x^1x^2. \tag{2.4}$$

Proposition 2.2 *The equilibrium states of the fractional Lagrange system* (2.1) *are given as the union of the following three families:*

$$E_1 := \{e_1^m = (m, 0, 0) \in \mathbf{R}^3 |\ m \in \mathbf{R}\}, \quad E_2 := \{e_2^m = (0, m, 0) \in \mathbf{R}^3 |\ m \in \mathbf{R}\},$$
$$E_3 := \{e_3^m = (0, 0, m) \in \mathbf{R}^3 |\ m \in \mathbf{R}\}.$$

Proof. The equilibrium states are solutions of the equations $f_i(x) = 0, i = \overline{1,3}$ where $f_i,\ i = \overline{1,3}$ are given by (2.2). □

Let us we present the study of asymptotic stability of equilibrium states for the fractional system (2.1). Finally, we will discuss how to stabilize the unstable equilibrium states of the system (3.1) via fractional order derivative. For this study we apply the Matignon's test [19].

The Jacobian matrix associated to system (2.1) is:

$$J(x) = \begin{pmatrix} 0 & x^3 & x^2 \\ x^3 & 0 & x^1 \\ x^2 & x^1 & 0 \end{pmatrix}.$$

Proposition 2.3 ([19]) *Let x_e be an equilibrium state of system* (2.1) *and $J(x_e)$ be the Jacobian matrix $J(x)$ evaluated at x_e. Then x_e is locally asymptotically stable, iff all eigenvalues of the matrix $J(x_e)$ satisfy the inequality:* $|arg(\lambda(J_(x_e)))| > \dfrac{q\pi}{2}.$ □

Proposition 2.4 *The equilibrium states $e_i^m \in E_i,\ i = \overline{1,3}$ are unstable* $(\forall) q \in (0, 1)$.

Proof. The characteristic polynomial of the matrix $J(e_1^m) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m \\ 0 & m & 0 \end{pmatrix}$ is $p_{J(e_1^m)}(\lambda) = \det(J(e_1^m) - \lambda I) = -\lambda(\lambda^2 - m^2)$. For $m = 0$, the characteristic polynomials of $J(e_0)$ is $p_{J(e_0)}(\lambda) = \lambda^3$.

The characteristic polynomials of $J(e_i^m),\ i = 2, 3$ are $p_{J(e_i^m)}(\lambda) = -\lambda(\lambda^2 - m^2)$.

The equations $p_{J(e_0)}(\lambda) = 0$ and $p_{J(e_i^m)}(\lambda) = 0$ for $i = \overline{1,3}$ have the root $\lambda_1 = 0$. Since $arg(\lambda_1) = 0 < \frac{q\pi}{2}$ for all $q \in (0, 1)$, by Proposition 2.3 follows that the equilibrium states e_0 and $e_i^m, i = \overline{1,3}$ are unstable for all $q \in (0, 1)$. □

3 Stability analysis of the fractional-order Lagrange system with controls (3.3)

In the case when x_e is a unstable equilibrium state of the fractional-order system (2.1), we associate to (2.1) a new fractional system, called the *fractional-order Lagrange system with controls* and given by:

$$\begin{cases} D_t^q x^1(t) &=& x^2(t)x^3(t)+u_1(t) \\ D_t^q x^2(t) &=& x^1(t)x^3(t)+u_2(t), \\ D_t^q x^3(t) &=& x^1(t)x^2(t)+u_3(t), \end{cases} \qquad q\in(0,1), \qquad (3.1)$$

where $u_i(t), i=\overline{1,3}$ are control functions.

In this section we take the control functions $u_i(t), i=\overline{1,3}$, given by:

$$u_1(t)=c_1x^1(t), \quad u_2(t)=c_2x^2(t), \quad u_3(t)=c_2x^3(t), \quad c_1,c_2\in\mathbf{R}. \qquad (3.2)$$

With the control functions (3.2), the system (3.1) becomes:

$$\begin{cases} D_t^q x^1(t) &=& x^2(t)x^3(t)+c_1x^1(t) \\ D_t^q x^2(t) &=& x^1(t)x^3(t)+c_2x^2(t), \\ D_t^q x^3(t) &=& x^1(t)x^2(t)+c_2x^3(t) \end{cases} \qquad q\in(0,1), \qquad (3.3)$$

where $c_1,c_2\in\mathbf{R}^*$ are real constants.

The system (3.3) is called the *fractional-order Lagrange system with controls* c_1,c_2.

If one selects the parameters c_1,c_2 which then make the eigenvalues of the Jacobian matrix of fractional model (3.3) satisfy one of the conditions from Proposition 2.3, then its trajectories asymptotically approaches the unstable equilibrium state x_e in the sense that $\lim_{t\to\infty}\|x(t)-x_e\|=0$, where $\|\cdot\|$ is the Euclidean norm.

The Jacobian matrix of the fractional-order Lagrange system (3.3) with controls c_1,c_2 is:

$$J(x,c_1,c_2)=\begin{pmatrix} c_1 & x^3 & x^2 \\ x^3 & c_2 & x^1 \\ x^2 & x^1 & c_2 \end{pmatrix}.$$

Proposition 3.1 *Let be the fractional-order Lagrange system* (3.3).
(i) If $c_1<0, c_2<0$, *then* e_0 *is asymptotically stable* $(\forall)\ q\in(0,1)$;
(ii) If $c_1>0, c_2>0$ *or* $c_1c_2<0$, *then* e_0 *is unstable* $(\forall)\ q\in(0,1)$.

Proof. The characteristic polynomial of $J(e_0,c_1,c_2)$ is $p_0(\lambda)=-(\lambda-c_1)(\lambda-c_2)^2$. The roots of the equation $p_0(\lambda)=0$ are $\lambda_1=c_1,\ \lambda_{2,3}=c_2$.
(i) We suppose $c_1<0$ and $c_2<0$. In this case we have $Re(\lambda_i)<0$ for $i=\overline{1,3}$. Since $|arg(\lambda_i)|=\pi>\frac{q\pi}{2}, i=\overline{1,3}$ for all $q\in(0,1)$, by Proposition 2.3, it implies that e_0^m is locally asymptotically stable.
(ii) We suppose $c_1>0, c_2>0$ or $c_1c_2<0$. Since $J(e_0,c_1,c_2)$ has at least a positive eigenvalue, it follows that e_0 is unstable. Hence, (i) and (ii) hold. □

Proposition 3.2 *Let be the fractional-order Lagrange system* (3.3) *and* $q\in(0,1)$.
(i) If $c_1<0, c_2<0$, *then* e_1^m *is asymptotically stable* $(\forall)m\in(c_2,-c_2)$ *and unstable* $(\forall)\ m\in(-\infty,c_2)\cup(-c_2,\infty)$.
(ii) If $c_1>0, c_2>0$ *or* $c_1c_2<0$, *then* e_1^m *is unstable* $(\forall)m\in\mathbf{R}^*$.

Proof. The Jacobian matrix of (3.3) at e_1^m is $J(e_1^m, c_1, c_2) = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & m \\ 0 & m & c_2 \end{pmatrix}$ whose characteristic polynomial is $p_1(\lambda) = -(\lambda - c_1)[(\lambda - c_2)^2 - m^2]$. The roots of the equation $p_1(\lambda) = 0$ are $\lambda_1 = c_1$, $\lambda_{2,3} = c_2 \pm m$.
(i) We suppose $c_1 > 0, c_2 > 0$. The eigenvalues λ_i, $i = \overline{1,3}$ are all negative if and only if $m \in (c_2, -c_2)$. Then e_1^m is asymptotically stable.
(ii) We suppose $c_1 > 0, c_2 > 0$ or $c_1 c_2 < 0$. In these cases,the matrix $J(e_0, c_1, c_2)$ has at least a positive eigenvalue. It follows that e_1^m is unstable. Hence the assertions (i) and (ii) hold. □

Proposition 3.3 *Let be the fractional Lagrange system* (3.3) *and* $q \in (0,1)$.
(i) *If* $c_1 < 0, c_2 < 0$, *then* e_2^m *and* e_3^m *are asymptotically stable for all* $m \in (-\sqrt{c_1 c_2}, \sqrt{c_1 c_2})$ *and unstable* $(\forall) m \in (-\infty, -\sqrt{c_1 c_2}) \cup (\sqrt{c_1 c_2}, \infty)$.
(ii) *If* $c_1 > 0$, $c_2 < 0$ *or* $c_2 > 0$, $c_1 \in \mathbf{R}$, *then* e_2^m *and* e_3^m *are unstable* $(\forall) m \in \mathbf{R}^*$.

Proof. We consider $k \in \{2, 3\}$. The Jacobian matrices of (3.3) at e_k^m are $J(e_2^m, c_1, c_2) = \begin{pmatrix} c_1 & 0 & m \\ 0 & c_2 & 0 \\ m & 0 & c_2 \end{pmatrix}$ and $J(e_3^m, c_1, c_2) = \begin{pmatrix} c_1 & m & 0 \\ m & c_2 & 0 \\ 0 & 0 & c_2 \end{pmatrix}$. Its characteristic polynomials are $p_k(\lambda) = -(\lambda - c_2)[\lambda^2 - (c_1 + c_2)\lambda + c_1 c_2 - m^2]$. The roots of the equation $p_k(\lambda) = 0$ are $\lambda_1 = c_2$, $\lambda_{2,3} = \frac{(c_1+c_2) \pm \sqrt{\Delta}}{2} \in \mathbf{R}$, where $\Delta = (c_1 - c_2)^2 + 4m^2$.
(i) We suppose $c_1 < 0, c_2 < 0$. We have $\lambda_2, \lambda_3 < 0$ if and only if $\lambda_2 + \lambda_3 < 0$ and $\lambda_2 \lambda_3 > 0$. Then $c_1 + c_2 < 0$ and $(c_1 + c_2)^2 - \Delta > 0$. It follows that $\lambda_i < 0, i = \overline{1,3}$ for all $m \in (-\sqrt{c_1 c_2}, \sqrt{c_1 c_2})$. Therefore, e_k^m is asymptotically stable.
(ii) We suppose $c_1 > 0$ and $c_2 < 0$ or $c_2 > 0$ and $c_1 \in \mathbf{R}$. In this cases, $J(e_k^m, c_1, c_2)$ has at least a positive eigenvalue and so e_k^m is unstable $(\forall) m \in \mathbf{R}^*$. Therefore, the assertions (i) and (ii) hold. □

Example 3.1 By choosing the control parameters c_1, c_2 that satisfy one condition from Proposition 3.3, then the trajectories of the fractional-order Lagrange system (3.3) are driven to the stable equilibrium point e_3^m $(m \neq 0)$. For example, we select $c_1 = -1.75, c_2 = -2$, then the stability condition (i) of Proposition 3.3 is achieved. This implies that, the trajectories of the system (3.3) converge to the equilibrium point $e_3 = (0, 0, m)$ when $m \in (-1.8708, 1.8708)$ and $q \in (0,1)$. For example, the fractional-order Lagrange system is asymptotically stable at $e_3 = (0, 0, 1.75)$ for $q \in (0,1)$. □

Using Matlab, in Table 3.1 we give a set of values for c_1, c_2, the equilibrium states and corresponding eigenvalues of fractional-order Lagrange system (3.3).

$c_1, c_2 \in \mathbf{R}^*$	*Eigenvalues*	$m,\ q$	e_i^m	*Stability*
$c_1 = -0.2,\ c_2 = -0.8$	$\lambda_1 = -0.2,\ \lambda_{2,3} = -0.8$	$q \in (0,1)$	e_0	*stable*
$c_1 = -2,\ c_2 = -1.85$	$\lambda_1 = -2,\ \lambda_2 = -0.85$ $\lambda_3 = -2.85$	$m = 1$ $q \in (0,1)$	e_1^m	*stable* $m \in (-1.85, 1.85)$
$c_1 = -7.2,\ c_2 = -0.2$	$\lambda_1 = -0.2,\ \lambda_2 = 0.4182$ $\lambda_3 = -7.8182$	$m = -2$ $q \in (0,1)$	e_2^m	*unstable* $m \in (-\infty, -1.2) \cup (1.2, \infty)$
$c_1 = -1.75,\ c_2 = -2$	$\lambda_1 = -2,\ \lambda_2 = -0.9911$ $\lambda_3 = -2.7588$	$m = 1.75$ $q \in (0,1)$	e_3^m	*stable* $m \in (-1.8708, 1.8708)$

Table 3.1. *The controls* c_1, c_2, *equilibrium states* e_i^m *and corresponding eigenvalues.*

4 Numerical integration of the fractional-order Lagrange system (3.3)

Consider the following general form of the initial value problem (IVP) with Caputo derivative [20]:

$$D_t^q y(t) = f(t, y(t)), \quad y(0) = y_0, \quad t \in I = [0, T],\ T > 0 \tag{4.1}$$

where $y : I \to \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous nonlinear function and $q \in (0,1)$, represents the order of the derivative.

The right-hand side of the IVP (4.1) in considered examples are Lipschitz functions and the numerical method used in this works to integrate system (4.1) is the *Fractional Euler's method.*

Since f is assumed to be continuous function, every solution of the initial value problem given by (4.1) is also a solution of the following *Volterra fractional integral equation*:

$$y(t) = y(0) + I_t^q f(t, y(t)), \tag{4.2}$$

where I_t^q is the $q-$order Riemann-Liouville integral operator (2.1).

$$I_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s))ds, \ q > 0, \quad s \in [0, T]. \tag{4.3}$$

Moreover, every solution of (4.2) is a solution of the (IVP) (4.1).

To integrate the fractional equation (4.1), means to find the solution of (4.2) over the interval $[0, T]$. In this context, a set of points $(t_j, y(t_j))$ are produced which are used as approximated values. In order to achieve this approximation, the interval $[0, T]$ is partitioned into n subintervals $[t_j, t_{j+1}]$ each equal width $h = \frac{T}{n}$, $t_j = jh$ for $j = 0, 1, ..., n$. For the fractional-order q and $j = 0, 1, 2, ...$, it computes an approximation denoted as y_{j+1} for $y(t_{j+1})$, $j = 0, 1, ...$

The general formula of the fractional Euler's method for to compute the elements y_j, is

$$y_{j+1} = y_j + \frac{h^q}{\Gamma(q+1)} f(t_j, y(t_j)), \qquad t_{j+1} = t_j + h, \quad j = 0, 1, ..., n. \tag{4.4}$$

For more details, see [20, 6, 2].

For the numerical integration of the system (3.3), we apply the fractional Euler method (FEM). For this, consider the following fractional differential equations

$$\begin{cases} D_t^q x^i(t) & = & F_i(x^1(t), x^2(t), x^3(t)), \quad i = \overline{1,3}, \quad t \in (t_0, \tau), \ q \in (0,1) \\ x(t_0) & = & (x^1(t_0), x^2(t_0), x^3(t_0)) \end{cases} \tag{4.5}$$

where

$$\begin{cases} F_1(x(t)) & = & x^2(t)x^3(t) + c_1 x^1(t), \\ F_2(x(t)) & = & x^1(t)x^3(t) + c_2 x^2(t), \qquad c_1, c_2 \in \mathbf{R}^*, \\ F_3(x(t)) & = & x^1(t)x^2(t) + c_2 x^3(t). \end{cases} \tag{4.6}$$

Since the functions $F_i(x(t)), i = \overline{1,3}$ are continuous, the initial value problem (4.5) is equivalent to system of Volterra integral equations, which is given as follows:

$$x^i(t) \ = \ x^i(0) + \ I_t^q F_i(x^1(t), x^2(t), x^3(t)), \qquad i = \overline{1,3}. \tag{4.7}$$

The system (4.7) is called the *Volterra integral equations associated to fractional-order Lagrange system* (4.5).

The problem for solving the system (4.5) is reduced to one of solving a sequence of systems of fractional equations in increasing dimension on successive intervals $[j, (j+1)]$.

For the numerical integration of the system (4.6) one can use the fractional Euler method (the formula (4.4)), which is expressed as follows:

$$x^i(j+1) = x^i(j) + \frac{h^q}{\Gamma(q+1)} F_i(x^1(j), x^2(j), x^3(j)), \quad i = \overline{1,3}, \tag{4.8}$$

where $j = 0, 1, 2, ..., N, h = \dfrac{T}{N}, T > 0, N > 0.$

More precisely, the numerical integration of the fractional system (4.5) is given by:

$$\begin{cases} x^1(j+1) & = x^1(j) + h^q \ \dfrac{1}{\Gamma(q+1)} (x^2(j)x^3(j) + c_1 x^1(j)) \\ x^2(j+1) & = x^2(j) + h^q \ \dfrac{1}{\Gamma(q+1)} (x^1(j)x^3(j) + c_2 x^2(j)) \\ x^3(j+1) & = x^3(j) + h^q \ \dfrac{1}{\Gamma(q+1)} (x^1(j)x^2(j) + c_2 x^3(j)) \\ x^i(0) & = x_e^i + \varepsilon, \quad i = \overline{1,3}. \end{cases} \tag{4.9}$$

Using [20, 8], we have that the numerical algorithm given by (4.9) is convergent.

Example 4.1 Let us we present the numerical simulation of solutions of the fractional-order Lagrange system (3.3) which has considered in Example 3.1.
For this we apply the algorithm (4.9) and software Maple. Then, in (4.9) we take: $c_1 = -1.75, c_2 = -2,\ h = 0.01, \varepsilon = 0.01, N = 500, t = 502$.

The orbits $(n, x^i(n)), i = \overline{1,3}$ for the solutions of fractional-order Lagrange system for the equilibrium state $e_3 = (0, 0, 1.75)$ have the representations given in figures Fig. 1(a), 2(a), 3(a) (for $q = 0.65$) and Fig. 1(b), 2(b), 3(b) (for $q = 1$). □

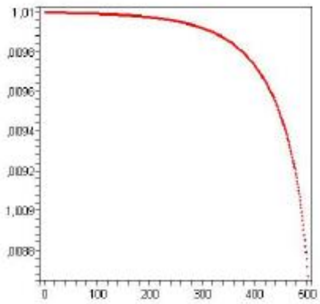

Fig.1.(a) $(n, x^1(n))$ for $q = 0.65$

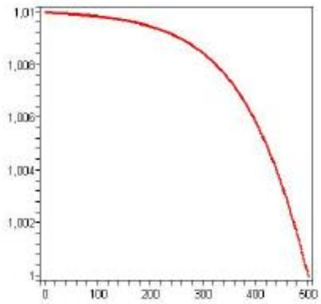

Fig.1.(b) $(n, x^1(n))$ for $q = 1$

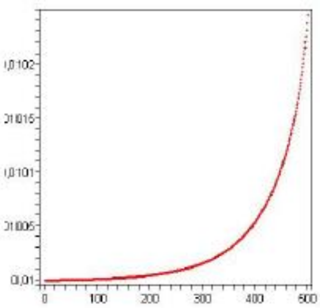

Fig.2.(a) $(n, x^2(n))$ for $q = 0.65$

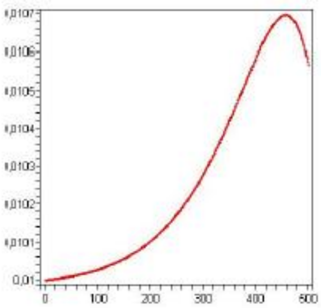

Fig.2.(b) $(n, x^2(n))$ for $q = 1$

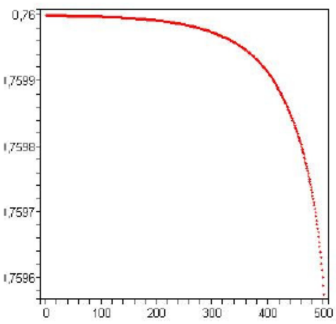

Fig.3.(a) $(n, x^3(n))$ for $q = 0.65$

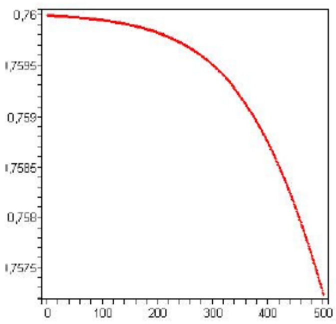

Fig.3.(b) $(n, x^3(n))$ for $q = 1$

The numerical simulations confirm the validity of the theoretical analysis.

Remark 4.1 Appyling (4.9) and Maple for the numerical simulation of solutions of fractional Lagrange system (3.3) for each pair (c_1, c_2) of values given in the Table (3.1), it will be found that the results obtained are valid. □

Conclusions. This paper presents the fractional-order Lagrange system (3.3) associated to system (1.1). The fractional-order Lagrange system (3.3) was studied from fractional differential equations theory point of view: asymptotic stability, determining of sufficient conditions on parameters c_1, c_2 to control the chaos in the proposed fractional system and numerical integration of the fractional model (4.2). By choosing the right parameters c_1 and c_2 in the fractional model (3.3), this work offers a series of chaotic fractional differential systems. □

Acknowledgments. The author has very grateful to be reviewers for their comments and suggestions.

Conflict of interest: The author declares no conflict of interest.

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