

Chromatic Number using Complete Graph

Kolappan Velayutham^{1*}, Winson Rajaian², Sulaiman Nazarudeen³,
Santhosh Moni⁴

¹Department of Information Technology, University of Technology and Applied Sciences, Muscat, Oman.
kolappan.velayutham@utas.edu.om

²Department of Information Technology, University of Technology and Applied Sciences, Muscat, Oman.
winson.rajaian@utas.edu.om

³Department of Information Technology, University of Technology and Applied Sciences, Muscat, Oman.
sulaiman.rawther@utas.edu.om

⁴Department of Information Technology, University of Technology and Applied Sciences, Muscat, Oman.
santhosh.moni@utas.edu.om

*Corresponding Author: kolappan.velayutham@utas.edu.om

ABSTRACT: A proper coloring of a graph requires the assignment of colors to the vertices of the graph in such a way that no two adjacent vertices are assigned the same color. This requirement leads to the construction of a complete graph whose vertices represent the color classes of the proper coloring. We call this graph as Color Class Complete Graph (CCCG). Finding the chromatic number of a graph requires the construction of the smallest CCC graph. In this paper, we find the chromatic number of the p -petal graph, when $p = 3$, using the CCC Graph. We also find the chromatic number of the line graph, middle graph and the total graph of the 3-petal graph.

KEY WORDS: proper coloring, chromatic number, p -petal graph, line graph, middle graph, total graph

Date of Submission: 15-05-2024

Date of acceptance: 29-05-2024

I. INTRODUCTION

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_i: i = 1 \text{ to } n\}$ and edge set $E = \{e_i: i = 1 \text{ to } m\}$. A proper coloring of the vertices of G is the assignment of k colors from the set $C = \{c_i: i = 1 \text{ to } k\}$ such that no two adjacent vertices are assigned the same color. The chromatic number of the graph G is $\chi(G) = k$, the smallest possible integer.

A proper coloring of G partitions its vertex set into a color class set $C = \{C_i: i = 1 \text{ to } k\}$ of disjoint color classes. Brooks¹ defined the chromatic number of G , denoted $\chi(G)$ as the minimum cardinality of C , that is the smallest possible value of k . For the smallest value of k , it is required that at least one vertex in C_i is adjacent to at least one vertex in C_j for $i \neq j$. This requirement leads to the construction of the smallest possible complete graph in which each vertex is represented by a color class, and we call it the color class complete graph (CCCG).

The chromatic number of the Petersen graph is 3. We can construct the three vertex color class complete graph K_3 with color classes

1. $C_1 = \{v_1, v_3, v_7\}$
2. $C_2 = \{v_2, v_4, v_6, v_{10}\}$
3. $C_3 = \{v_5, v_8, v_9\}$

Refer to Figure 1.

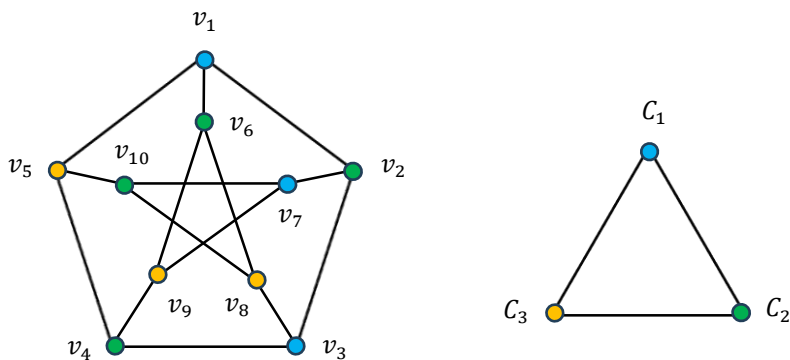


Figure 1: Petersen graph and its color class complete graph K_3

A petal graph is a connected graph G with maximum degree three, minimum degree two, and such that the set of vertices of degree three induces a 2-regular graph and the set of vertices of degree two induces an empty graph². A petal graph G of n vertices and a petals with petal sequence $\{P_j\}$ is said to be a p -petal graph³, denoted $G = P_{n,p}$, if every petal in G is of size p and $l(P_i, P_{i+1}) = 2, i = 0, 1, 2, \dots, a - 1$ with $P_{a+1} = P_0$. In a p -petal graph the petal size p is always odd. In this paper, we consider 3-petal graphs with $a \geq 3$ petals. Figure 2 shows two different versions of the 3-petal graph with the same number of petals. The 3-petal graphs are planar if the number of petals is even. The p -petal graph with the a number of petals has $3a$ vertices and $4a$ edges.

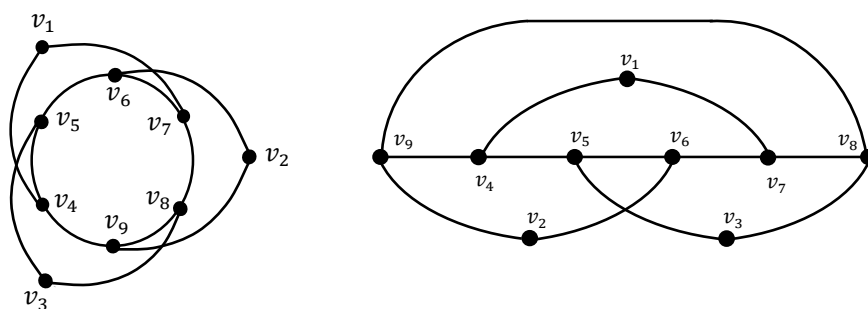


Figure 2: Two versions of a 3-petal graph with 3 petals

In the works of Whitney⁴ originated the concept of line graph. Consider an undirected graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, \dots, v_{n-1}, v_n\}$ and $E = \{e_1, e_2, e_3, \dots, e_{m-1}, e_m\}$. The line graph of G is the graph $L(G)$ in which each vertex in $L(G)$ represents an edge e_i in G . Two vertices in $L(G)$ are connected by an edge if their corresponding edges in G are incident with a common vertex in G .

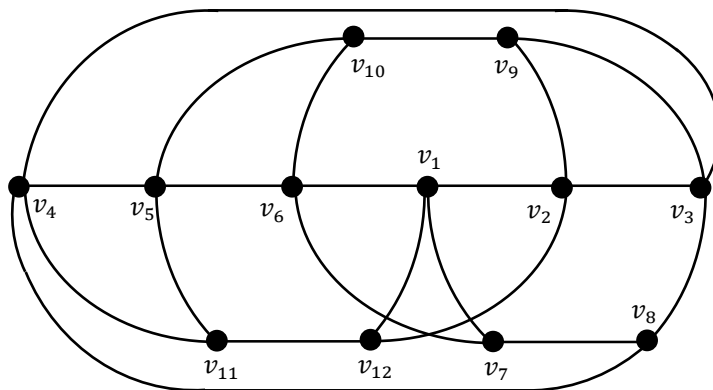


Figure 3: The line graph $L(G)$ of a 3-petal graph with 3 petals

Consider a p -petal graph $G = P_{n,p}$ with number of petals a . This graph G has $3a$ vertices and $4a$ edges. The line graph of the p -petal graph $L(G)$ is clearly not another p -petal graph, because it fails to satisfy the required conditions for a petal graph. The maximum degree of the vertices of the line graph is 4 and the minimum degree is 3. There are $4a$ vertices and $7a$ edges in $L(G)$. Figure 3 shows the line graph of the 3-petal graph with 3 petals. The p -petal graph has 9 vertices and 12 edges, whereas its line graph has 12 vertices and 21 edges.

T. Hamada and I. Yoshimura⁵ introduced the middle graph of a graph G , denoted $M(G)$ as constructed in such §1a way that the vertex set of $M(G)$ is the union of the vertex set and the edge set, that is $V(M(G)) = V(G) \cup E(G)$. Two vertices of $M(G)$ are adjacent if either

- (i) their corresponding edges are incident with the same vertex in G or
- (ii) their corresponding vertex and edge are incident in G .

The middle graph $M(G)$ of the p -petal graph is clearly not another p -petal graph. The maximum degree of the vertices of the middle graph is 6 and the minimum degree is 2. There are $7a$ vertices, of which $2a$ vertices are of degree six, $2a$ vertices of degree five, $2a$ vertices of degree three and a vertices of degree two. There are $15a$ edges in $M(G)$. Figure 4 shows the middle graph of the 3-petal graph with 3 petals. The core cycle of the middle graph has $2a$ vertices of degree six and $2a$ vertices of degree three.

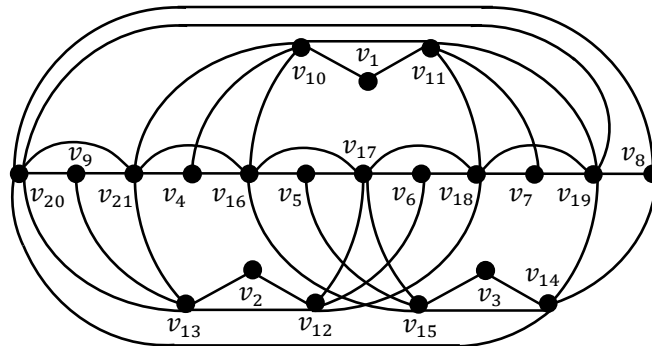


Figure 4: The middle graph $M(G)$ of a 3-petal graph with 3 petals

Mehdi Behzad⁶ defined the total graph of a graph. The total graph $T(G)$ of a petal graph G is one whose vertex set $V(T(G)) = V(G) \cup E(G)$. Two vertices are adjacent in $T(G)$ if either

- (i) their corresponding vertices in G are adjacent in G or
- (ii) their corresponding edges are incident with the same vertex in G or
- (iii) their corresponding vertex and edge are incident in G .

The total graph $T(G)$ of the p -petal graph is clearly not another p -petal graph. The maximum degree of the vertices of the total graph is 6 and the minimum degree is 4. There are $7a$ vertices, of which $4a$ vertices are of degree six, $2a$ vertices of degree five and a vertices of degree four. There are $19a$ edges in $T(G)$. Figure 5 shows the total graph of the 3-petal graph with 3 petals. The vertices of degree six form the core cycle of the total graph.

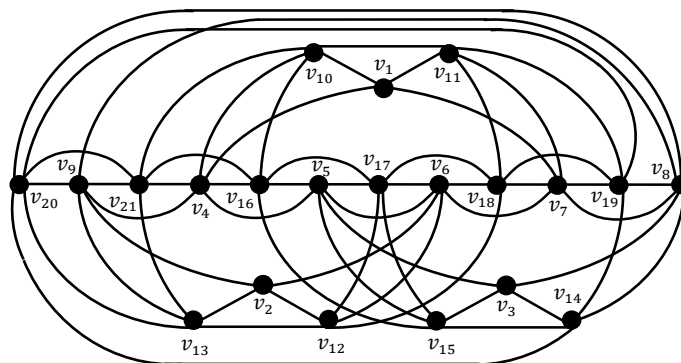


Figure 5: The total graph $T(G)$ of a 3-petal graph with 3 petals

II. MAIN RESULTS

In this paper, we consider the 3-petal graphs only. V. Kolappan⁷ has already proved the first theorem using another method. We find these results using the complete graph in this work.

1. The chromatic number of a 3-petal graph is three.
2. The chromatic number of the line graph of a 3-petal graph three if a is even and four if a is odd.
3. The chromatic number of the middle graph of a 3-petal graph is four.
4. The chromatic number of the total graph of a 3-petal graph is four.

Theorem 1

The chromatic number of the 3-petal graph $G = P_{n,3}$ is given by $\chi(G) = 3$

Let $G = P_{n,3}$ be a 3-petal graph with a petals. There will be $3a$ vertices in the vertex set $V = \{v_1, v_2, v_3, \dots, v_{3a-2}, v_{3a-1}, v_{3a}\}$ of the 3-petal graph. The maximum degree of the vertices of a 3-petal graph is 3. The core of the 3-petal graph is an even cycle, irrespective of the number of petals being even or odd. Every petal starts from an odd suffixed vertex and ends at the even suffixed vertex. From the construction of the p -petal graph, every odd suffixed vertex is adjacent to two even suffixed vertices and a central vertex, every even suffixed vertex is adjacent to two odd suffixed vertices and a central vertex, and every central vertex is adjacent to an odd and an even vertex. The petals form the vertex group $A = \{v_1, v_2, v_3, \dots, v_a\}$ and the core cycle of the petal graph forms the vertex group $B = \{v_{a+1}, v_{a+2}, v_{a+3}, \dots, v_{3a}\}$.

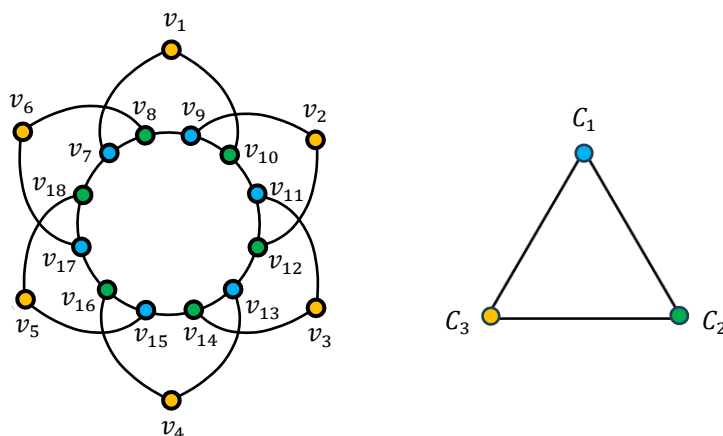


Figure 6: The chromatic number of the 3-petal graph with 6 petals is 3

We can use the complete graph K_3 with its three vertices represented by the three color classes C_1 , C_2 and C_3 . Refer Figure 6. Let the color class C_1 contains each of the odd suffixed vertices from B , the color class C_2 contains each of the even suffixed vertices from B and the color class C_3 contains each of the central vertices from B that is adjacent to an even vertex and an odd vertex.

- (i) $C_1 = \{v_7, v_9, v_{11}, v_{13}, v_{15}, v_{17}\}$
- (ii) $C_2 = \{v_8, v_{10}, v_{12}, v_{14}, v_{16}, v_{18}\}$
- (iii) $C_3 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

This color class complete graph K_3 represents the proper coloring of the 3-petal graph G and fixes its chromatic number to be three. ■

Theorem 2

The chromatic number of the line graph of the 3-petal graph $G = P_{n,3}$ is given by $\chi(L(G)) = \begin{cases} 3, & a \text{ is even} \\ 4, & a \text{ is odd} \end{cases}$

Consider the 3-petal graph $G = P_{n,3}$ with a petals. The line graph of the 3-petal graph $L(G)$ has $2a$ vertices $v_1, v_2, v_3, \dots, v_{2a-1}$ and v_{2a} of degree four that form the core of the graph, and $2a$ vertices $v_{2a+1}, v_{2a+2}, v_{2a+3}, \dots, v_{4a-1}$ and v_{4a} of degree three that lie on the exterior to the core of the graph as depicted in Figures 7 and 8. We have $V(G) = A \cup B$, where $A = \{v_1, v_2, v_3, \dots, v_{2a}\}$ and $B = \{v_{2a+1}, v_{2a+2}, v_{2a+3}, \dots, v_{4a}\}$.

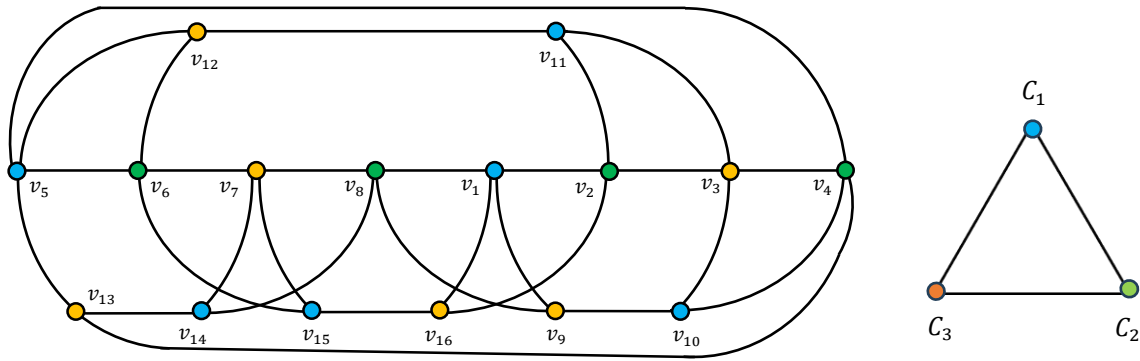


Figure 7: The chromatic number of the line graph $L(G)$ with even petals is three

The vertices v_1, v_2 and v_{4a} form a complete graph K_3 as an induced subgraph. This forces the minimum number of colors required for a proper coloring of $L(G)$ to be three.

Case 1: The number of petals a is even

Consider the complete graph K_3 with color classes C_1, C_2 and C_3 . The $4a$ vertices of $L(G)$ are included in the color classes as follows:

1. $C_1 = A_1 \cup B_1 = \{v_1, v_5, \dots, v_{2a-3}\} \cup \{v_{2a+2}, v_{2a+3}, v_{2a+6}, v_{2a+7}, \dots, v_{4a-5}, v_{4a-2}, v_{4a-1}\}$
2. $C_2 = A_2 \cup B_2 = \{v_2, v_4, \dots, v_{2a}\}$, where $B_2 = \emptyset$
3. $C_3 = A_3 \cup B_3 = \{v_3, v_7, \dots, v_{2a-1}\} \cup \{v_{2a+1}, v_{2a+4}, v_{2a+5}, \dots, v_{4a-4}, v_{4a-3}, v_{4a}\}$

Refer Figure 7.

Thus, the chromatic number of the line graph $L(G)$ of the 3-petal graph G with even number of petals is three.

Case 2: The number of petals a is odd

Consider the complete graph K_3 with color classes C_1, C_2 and C_3 . The $4a$ vertices of $L(G)$ are included in the color classes as follows:

1. $C_1 = A_1 \cup B_1 = \{v_1, v_5, \dots, v_{2a-1}\} \cup \{v_{2a+2}, v_{2a+3}, v_{2a+6}, v_{2a+7}, \dots, v_{4a-4}, v_{4a-3}\}$
2. $C_2 = A_2 \cup B_2 = \{v_2, v_4, \dots, v_{2a-2}, v_{2a}\}$, where $B_2 = \emptyset$
3. $C_3 = A_3 \cup B_3 = \{v_3, v_7, \dots, v_{2a-3}\} \cup \{v_{2a+1}, v_{2a+4}, v_{2a+5}, v_{2a+8}, v_{2a+9}, \dots, v_{4a-2}, v_{4a-1}\}$

The vertex v_{4a} is adjacent to v_1, v_2 and v_{4a-1} , each of which belongs to the three color classes C_1, C_2 and C_3 , so needs the fourth color class C_4 , thus extends K_3 into K_4 . Hence, the fourth color class is given by

4. $C_4 = \{v_{4a}\}$

Refer Figure 8.

Thus, the chromatic number of the line graph of the 3-petal graph G with odd number of petals is four. ■

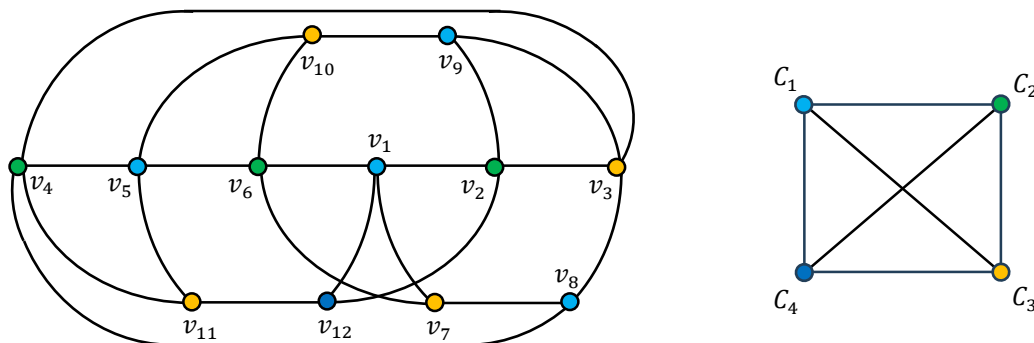


Figure 8: The chromatic number of the line graph $L(G)$ with even petals is four

Theorem 3

The chromatic number of the middle graph of the 3-petal graph $G = P_{n,3}$ is given by $\chi(M(G)) = 4$

Consider the 3-petal graph $G = P_{n,3}$ with a petals. The middle graph of the 3-petal graph $M(G)$ has $2a$ vertices $v_{5a+1}, v_{5a+2}, v_{5a+3}, \dots, v_{7a-1}$ and v_{7a} of degree six; $2a$ vertices $v_{3a+1}, v_{3a+2}, v_{3a+3}, \dots, v_{5a-1}$ and v_{5a} of degree five; $2a$ vertices $v_{a+1}, v_{a+2}, v_{a+3}, \dots, v_{3a-1}$ and v_{3a} of degree three and a vertices v_1, v_2, \dots, v_a of degree two. The vertices of degree six form the core cycle of the middle graph.

Consider the central vertex v_1 of minimum degree. It forms a K_3 together with v_{3a+1} and v_{3a+2} . So, the chromatic number is at least three. The neighbouring vertices v_{3a+1} and v_{3a+2} of v_1 are of degree five.

Consider the vertex v_{3a+1} . This vertex forms a K_4 together with v_{a+1}, v_{5a+1} and v_{7a} . So, the chromatic number is at least four. The neighbouring vertices v_{5a+1} and v_{7a} of v_{3a+1} are of degree six. Consider the vertex v_{5a+1} . This vertex fails to form a K_5 together with its neighbouring vertices $v_1, v_{a+1}, v_{3a+2}, v_{5a+1}$ and v_{7a} .

This argument is true for all central vertices $v_i, i = 1$ to a . This forces the minimum number of colors required for a proper coloring of $M(G)$ to be four. This prevents the possibility of the chromatic number being five. Hence, the chromatic number is four.

Consider the complete graph K_4 with color classes C_1, C_2, C_3 and C_4 . The $7a$ vertices of $M(G)$ are included in the color classes as follows:

1. $C_1 = \{v_{a+1}, v_{a+3}, v_{a+5}, \dots, v_{3a-1}, v_{3a+2}, v_{3a+4}, v_{3a+6}, \dots, v_{5a}\}$
2. $C_2 = \{v_1, v_2, v_3, \dots, v_a, v_{5a+1}, v_{5a+3}, \dots, v_{7a-1}\}$
3. $C_3 = \{v_{a+2}, v_{a+4}, v_{a+6}, \dots, v_{3a}, v_{3a+1}, v_{3a+3}, \dots, v_{5a-1}\}$
4. $C_4 = \{v_{5a+2}, v_{5a+4}, v_{5a+6}, \dots, v_{7a}\}$

Refer Figure 9.

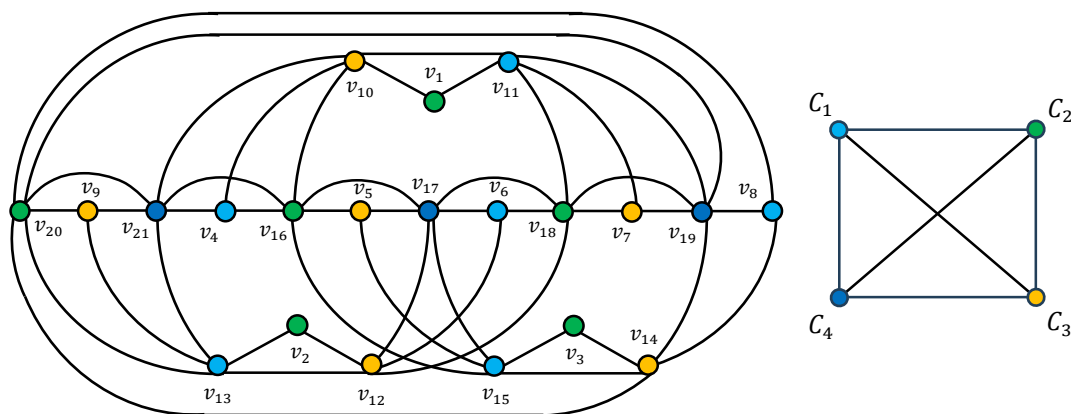


Figure 9: The chromatic number of the middle graph $M(G)$ is four

Thus, the chromatic number of the middle graph of the 3-petal graph G is four. ■

Theorem 4

The chromatic number of the total graph of the 3-petal graph $G = P_{n,3}$ is given by $\chi(T(G)) = 4$

Consider the 3-petal graph $G = P_{n,3}$ with a petals. The total graph $T(G)$ of the 3-petal graph has $4a$ vertices $v_{a+1}, v_{a+2}, v_{a+3}, \dots, v_{3a-1}, v_{3a}, v_{5a+1}, v_{5a+2}, v_{5a+3}, \dots, v_{7a-1}$ and v_{7a} of degree six; $2a$ vertices $v_{3a+1}, v_{3a+2}, v_{3a+3}, \dots, v_{5a-1}$ and v_{5a} of degree five; and a vertices v_1, v_2, \dots, v_a of degree four. The vertices of degree six form the core cycle of the total graph.

Consider the central vertex v_1 of minimum degree. It forms a K_3 together with v_{3a+1} and v_{3a+2} . So, the chromatic number is at least three. The neighbouring vertices v_{3a+1} and v_{3a+2} of v_1 are of degree five. Consider the vertex v_{3a+1} . This vertex forms a K_4 together with v_{a+1}, v_{5a+1} and v_{7a} . So, the chromatic number is at least four. The neighbouring vertices $v_{a+2}, v_{3a}, v_{5a+1}$, and v_{7a} of v_{a+1} are of degree six. Consider the vertex v_{a+2} . This vertex fails to form a K_5 together with its neighbouring vertices $v_a, v_{a+1}, v_{a+3}, v_{5a}, v_{5a+1}$ and v_{5a+2} .

This argument is true for all central vertices $v_i, i = 1$ to a . This forces the minimum number of colors required for a proper coloring of $T(G)$ to be four. This prevents the possibility of the chromatic number being five. Hence, the chromatic number is four.

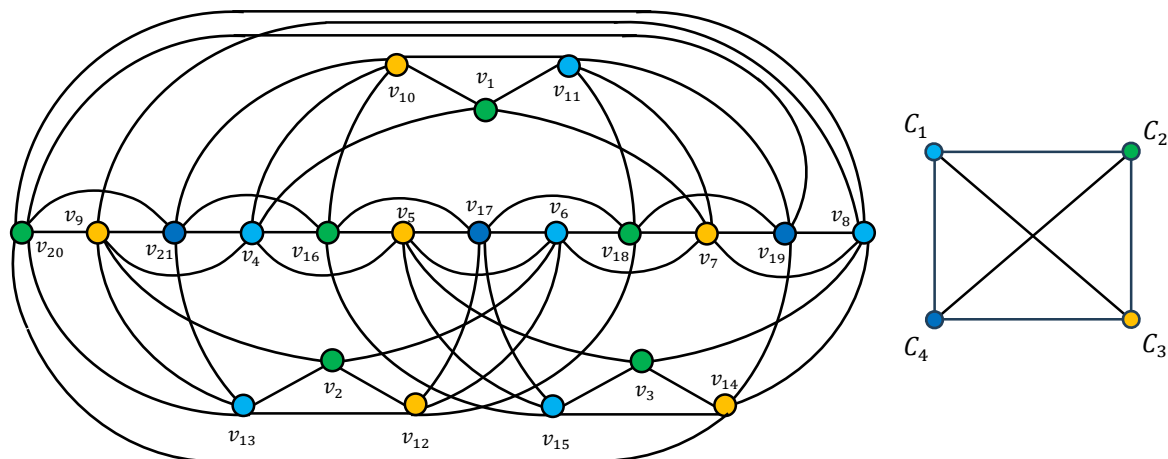


Figure 10: The chromatic number of the total graph $T(G)$ is four

Consider the complete graph K_4 with color classes C_1, C_2, C_3 and C_4 . The $7a$ vertices of $T(G)$ are included in the color classes as follows:

1. $C_1 = \{v_{a+1}, v_{a+3}, v_{a+5}, \dots, v_{3a-1}, v_{3a+2}, v_{3a+4}, v_{3a+6}, \dots, v_{5a}\}$
2. $C_2 = \{v_1, v_2, v_3, \dots, v_a, v_{5a+1}, v_{5a+3}, \dots, v_{7a-1}\}$
3. $C_3 = \{v_{a+2}, v_{a+4}, v_{a+6}, \dots, v_{3a}, v_{3a+1}, v_{3a+3}, \dots, v_{5a-1}\}$
4. $C_4 = \{v_{5a+2}, v_{5a+4}, v_{5a+6}, \dots, v_{7a}\}$

Refer Figure 10.

Thus, the chromatic number of the total graph of the 3-petal graph G is four. ■

III. CONCLUSIONS AND RECOMMENDATIONS

The 3-petal graph is particularly interesting because of its planarity properties. When $p > 3$ the p -petal graph is not planar. Its also interesting because of its feasibility in getting the results. We have still lot more variants of the p -petal graph and they are waiting for interesting results.

We hope that the following recommendations give important results:

1. Finding the b-chromatic number of the 3-petal graph, its line graph, middle graph and total graph.
2. Finding the chromatic and the b-chromatic number of graph such as the complementary graph, split graph and star graph that can be constructed from the 3-petal graph.
3. Finding the lower bounds for the chromatic number and the upper bounds b-chromatic number for the p -petal graphs, when $p > 3$.
4. There are many variants for the chromatic number. We can find these chromatic numbers for all the above graphs

REFERENCES

- [1]. Brooks, R. L. (1941). "On colouring the nodes of a network", Proc. Cambridge Philos. Soc., Volume 37, Number 2, pp: 194 - 197.
- [2]. David Cariolaro, Gianfranco Cariolaro. (2003). "Colouring the petals of a graph", The Electronic Journal of Combinatorics, Volume 10 (#R6), pp: 1 - 11.
- [3]. Kolappan, V., Selva Kumar, R. (2012). "Petal Graphs", International Journal of Pure and Applied Mathematics, Vol. 75, NO. 3, pp. 269 - 278.
- [4]. Hassler Whitney, (1932). "Congruent graphs and the connectivity of graphs", American Journal of Mathematics, Volume 54, Number 1, pp: 150 - 168.
- [5]. Hamada, T., Yoshimura, I. (1976). "Traversability and Connectivity of the middle graph of a graph", Discrete Mathematics, Volume 14, pp. 247 - 255.
- [6]. Behzad, M., Chartrand, G. (1966). "Total graphs and traversability" Proceedings of the Edinburgh Mathematical Society, pp. 117 - 120.
- [7]. Kolappan, V. (2014). "Study on Planarity of p -petal graphs", Ph. D. Thesis, p: 44.