A New Solution to Wave Equation in Inhomogeneous Plasmas with Bessel Function

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Abstract : *In this article the wave equation of an inhomogeneous plasma has been numerically solved by the use of Bessel function. Applying this mathematical approach produced the capability in comparison among other well known method such as WKB and Airy Function Method (AFM). The result which presented in this research shows that the Bessel Function Method (BFM) has reasonable agreement with the standard analytical solutions like AFM. Although there are some discrepancies between two methods, but in many ranges they can attain to common point of view. WKB method because of its simplicity and superabundant approximation could not satisfy the expected result especially at cutoff range, therefor is taken as a primary means to solve the mentioned problem.*

Keywords: *Airy function, cutoff, inhomogeneous, Debye sphere*

I. Introduction

The plasma state is a characterization of matter where long range electromagnetic interactions dominate the short range inter atomic or intermolecular forces among a large number of particles. Plasmas are generally high temperature entities, and some of the properties of a plasma are connected with thermal effects, and among the wave types we shall discuss are sound waves. Unmagnetized plasmas are generally the first to be studied because they are isotropic, i.e. the properties are the same in all directions. The waves that such a plasma will support are either high frequency electromagnetic waves which see the plasma as a simple dielectric due to the response of the electrons to the wave (by comparison the ions are generally immobile), or sound-type waves. We define the thermal speed to be the most probable speed in a Maxwellian distribution.

$$v_j = \sqrt{\frac{2kT_j}{m_j}} \quad j=e\,,\,i \qquad (1)$$

for electrons or ions. In addition to the thermal speeds for electrons and ions, however, there are two other fundamental parameters which characterize a plasma in the absence of a magnetic field, and these are the plasma frequency and the Debye length. The plasma frequency is the oscillation frequency of a simple unmagnetized plasma when the charge distribution is locally perturbed from its equilibrium, and is given for electrons by

$$\omega_{pe} = \sqrt{\frac{n_e e^2}{m_e \varepsilon_0}} \qquad (2)$$

The Debye length is the distance a thermal particle travels during a plasma period. Its definition is

$$\lambda_{De} = \sqrt{\frac{\varepsilon_0 k T_e}{n_e e^2}} = \frac{v_e}{\sqrt{2}\omega_{pe}} \qquad (3)$$

In fact, unless an assembly of charged particles is large enough that it is many Debye lengths in size, and of such a density that there are many particles in a Debye sphere, we do not call the assembly a plasma. It is thus apparent that plasma waves are fundamental to the very definition of a plasma [1]. In a cold plasma, these latter waves become a simple oscillation at the plasma frequency, below which the electromagnetic waves do not propagate. In a thermal plasma, however, there are sound-like waves near this frequency, and in a plasma with disparate electron and ion temperatures with $T_e >> T_i$, there is even a kind of hybrid sound wave that depends on the electron temperature and the ion mass. These waves even perturbed from its equilibrium, and is given for electrons by process. While these several kinds of behavior are already much more complicated than waves in ordinary fluids, they are very simple compared to the complexities added by a magnetic field. Even the kinds of nonlinearities in this simple plasma damp through a nondissipative are richer in both diversity and complexity than in fluid dynamics where only averages over the velocity distribution are analyzed. Compared to ordinary fluids, plasmas even have an additional kind of turbulence, called microturbulence, which has many kinds of sources.

The addition of magnetic field effects to the subject of plasma waves adds a host of new phenomena, among which are: anisotropy, since there is now a preferred direction; new kinds of transverse waves existing only in magnetized plasmas, which we call Alfv´en waves; finite Larmor orbit effects due to thermal motions about the magnetic field lines; and many other kinds of waves which are either totally new or greatly modified. Because of the anisotropy, the description of these effects is inevitably complicated algebraically, and this tends sometimes to obscure the physics. Even in a cold plasma where thermal effects are absent, the number of wave types added by the inclusion of the magnetic field is large, and wave types vary greatly with the angle of propagation with respect to the magnetic field. We find waves which are guided by the magnetic field in certain frequency ranges, and cases where the phase and group velocities are nearly normal to one another [2].
Whereas in a cold unmagnetized plasma, the only parameter that may lead to inhomogeneitiesis the plasma

density, the magnetic field not only adds a possible new source of inhomogeneity, but the gradients may appear in different directions.

When thermal effects are added to the cold plasma effects, the new phenomena can be grouped into two general categories: acoustic wave phenomena due to various kinds of sound waves; and kinetic phenomena due to the fact that in a thermal or near thermal distribution, there are some particles moving at or near the phase velocity. These particles have resonant interactions with the waves due to their long interaction time with the wave. These interactions can lead to either collisionless wave damping or to instabilities and wave growth. When coupled with magnetic field effects, finite Larmor orbit effects lead to even more new wave types and instabilities. The uniform plasmas of the previous chapters are idealizations which are rarely realized, although in some bounded regions, the approximation may be very good. We can generally assume a plasma is uniform if the plasma parameters vary little over a wavelength. In most laboratory plasmas, however, and over the vast regions of space, densities and magnetic fields vary to such an extent that it is sometimes difficult to estimate from uniform plasma theory where the wave energy will go and whether it will reach specified regions [3-5]. In many instances, these various effects can be separated and dealt with one at a time, but there are important cases where a combination of effects occur, several of which are simultaneously important, and each individual technique breaks down.

II. WKB method for one dimensional inhomogeneities

As an illustration of the method used in treating one-dimensional inhomogeneities, we consider an unmagnetized plasma with only a density variation. The wave equation,

$$\nabla(\nabla.E)-\nabla^2 E+\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}=-\mu_0\frac{\partial J}{\partial t} \quad (4)$$

depends only on variations in the plasma density through the plasma current, so we let $n_0 = n_0(x)$. Since all equilibrium quantities are independent of y and z, we can Fourier transform in those directions and assume harmonic time dependence. Then

$$\nabla \rightarrow \widehat{e_x}\frac{d}{dx}+\widehat{e_y}ik_y+\widehat{e_z}ik_z \qquad \frac{\partial}{\partial t}\rightarrow -i\omega$$

but another simplification is possible since the plasma is isotropic, since then one can rotate the coordinate system about the x-axis until the wave has no k_z component. Then the current may be derived from the equations of motion for ions and electrons

$$m_j\frac{\partial v_j}{\partial t}=q_j E \qquad j=i,e \quad (5)$$

$$j = en_0(x)(v_i-v_e) \quad (6)$$

with the result that

$$j \simeq \frac{in_0(x)e^2}{\omega m_e}E=\frac{i\varepsilon_0}{\omega}\omega_{pe}^2(x)E \quad (7)$$

Then the z-component of equation (4), which is not coupled the other component, is

$$-\left(\frac{d^2}{dx^2}-k_y^2\right)E_z+\left[\frac{\omega_{pe}^2(x)}{c^2}\right]E_z=0 \quad (8)$$

This can then be cast into the WKB form

$$\frac{d^2y}{dx^2}+k^2(x)y=0 \quad (9)$$

where y(x) = $E_z(x)$ and

$$k^2(x)\triangleq\frac{\omega^2-\omega_{pe}^2(x)}{c^2}-k_y^2 \quad (10)$$

For $k(x)$ = constant, the solution is trivial and of the form

$$y = A_1e^{ikx}+A_2e^{-ikx} \quad (11)$$

and represents waves traveling to the left and to the right. Looking for solutions which are similar to the uniform result, we assume an eikonal solution of the form

$$y(x) = A(x)e^{i\psi(x)} \quad (12)$$

where $A(x)$ is assumed to be a slowly varying amplitude and $\psi(x)$ is the eikonal, a rapidly varying phase such that $\psi'(x) = \pm k(x)$. In order to determine the limits of validity of this method, we choose the upper sign and insert equation (12) into equation (9), first noting the derivatives

$$y' = ikAe^{i\psi}+A'e^{i\psi}$$

$$y'' = -k^2Ae^{i\psi}+2ikA'e^{i\psi}+ik'Ae^{i\psi}+A''e^{i\psi} \quad (13)$$

so that equation (9) becomes

$$A''+2ikA'+ik'A = 0 \quad (14)$$

If A'' is assumed small, then to lowest order, A(x) = $\frac{1}{\sqrt{k(x)}}$, and the complete solution is written (for either sign) as

$$y(x) = \frac{A_0}{\sqrt{k(x)}}\exp\left[\pm i\int k(x')dx'\right] \quad (15)$$

Taking this as the zero order result, we assume the solution is modified by the correction term,

$$A(x) = \frac{[1+\eta(x)]}{\sqrt{k(x)}} \quad (16)$$

where again we assume that η is slowly varying so that we neglect η'' when this is inserted into equation (14). The result may be expressed as

$$\frac{\eta'}{1+\eta}=\frac{1}{4i}\frac{\frac{k''}{k^2}-\frac{3}{2}\frac{k''}{k^3}}{1+\frac{1}{2i}\frac{k'}{k^2}}$$

$$\eta\simeq\frac{1}{4i}\frac{k'}{k^2} \quad (17)$$

The condition for validity may thus be written as | η | << 1 or as

$$\left|\frac{1}{k}\frac{dk}{dx}\right|\ll k \quad (18)$$

which means that the change of wavelength over a wavelength should be small. This approximation fails when $k\rightarrow 0$ or when $k'\rightarrow\infty$, or whenever the wave approaches either a cutoff or a resonance. In figure.1 equation (15) which is the answer of problem is plotted with estimated values. In this figure y(x) is considered to have linear variation. It is shown that the answer is increased at first and after that it coming down. It is because of inhomogeneity in plasma and variety of density.

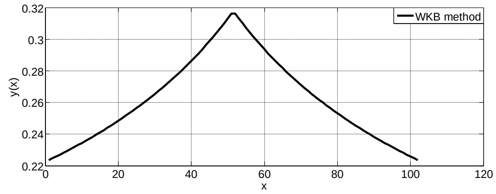

figure.1 y(x) which here is E_z(x) versus x

III. Airy Function Method

The behavior near a cutoff is important enough to justify further analysis, and was analyzed so long ago that it is considered by many to be part of the WKB formalism rather than restricted to the condition of equation (18). In the neighborhood of the cutoff, we expand $k^2(x)$ about the cutoff,

$$k^2(x) = k^2(x_0) + \frac{d}{dx}k^2(x)|_{x_0}(x-x_0) + O(x-x_0)^2 \quad (19)$$

where $k^2(x_0)$ = 0 defines x_0 and we define the coefficient such that $k^2(x) = \beta^2(x-x_0)$.Then the result of equation (15) valid whenever $|x-x_0| >> \beta^{-2/3}$. The behavior near the cutoff must come from the solution of the differential equation [4-8], however, since the approximations always fail sufficiently close to cutoff. The differential equation may be written as

$$\frac{d^2y}{dx^2} + \beta^2(x-x_0)y = 0 \quad (20)$$

which by means of the variable change $z \triangleq -\beta^{2/3}(x\ )$ can be written

$$y'' - zy = 0 \quad (21)$$

which is the Airy equation. The solutions of the Airy equation are well known [8] and may be represented by the two independent solutions

$$y(z) = c_1Ai(z) + c_2Bi(z) \quad (22)$$

which have the asymptotic forms

$$Ai(z) = \frac{1}{2\sqrt{\pi}z^{1/4}}e^{-\zeta}$$

$$Ai(-z) = \frac{1}{\sqrt{\pi}z^{1/4}}\left[\sin\left(\zeta+\frac{\pi}{4}\right) - \cos\left(\zeta+\frac{\pi}{4}\right)\right]$$

$$Bi(z) = \frac{1}{\sqrt{\pi}z^{1/4}}e^{\zeta}$$

$$Bi(-z) = \frac{1}{\sqrt{\pi}z^{1/4}}\left[\sin\left(\zeta+\frac{\pi}{4}\right) + \cos\left(\zeta+\frac{\pi}{4}\right)\right]$$

(23)

where $\zeta = \frac{2}{3}z^{3/2}$. These asymptotic solutions must be matched to the approximate eikonal solutions which represent incoming and outgoing waves, which given by equation (15)

$$y(x) = \frac{A_1}{(x-x_0)^{1/4}}e^{i\frac{2}{3}\beta(x-x_0)^{3/2}} + \frac{A_2}{(x-x_0)^{1/4}}e^{-i\frac{2}{3}\beta(x-x_0)^{3/2}} \quad x > x_0$$

$$y(x) = \frac{B_1}{|x-x_0|^{1/4}}e^{i\frac{2}{3}\beta|x-x_0|^{3/2}} + \frac{B_2}{|x-x_0|^{1/4}}e^{-i\frac{2}{3}\beta|x-x_0|^{3/2}} \quad x < x_0$$

(24)

In figure.2 the solution of problem is plotted. In this figure y(x) increases at first because of inhomogeneity and decrease later. It is observed that there is a sharp peak at the cutoff region.

IV. solution with Bessel function

The matching implied here between the asymptotic forms of the Airy function solutions and the WKB solutions requires that there be a finite region of overlap where both approximations are simultaneously valid. The conditions for validity are shown in figure.3 . When the exact expression for $k^2(x)$ is linear, the overlapping region is unbounded of course, but for any other variation, there is some limit when the linear approximation fails. If the real variation of $k^2(x)$ deviates substantially from linear before the WKB expressions are valid, then there may be no overlap, so that accurate possible.

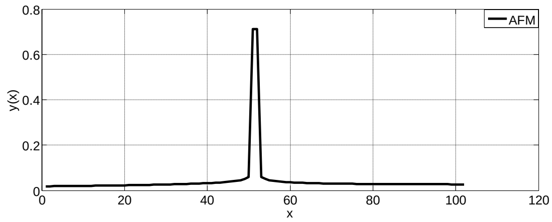

Figure. 2 y(x) which is (x) versus x in Airy function method

In this paper a solution of problem with Modified Bessel function is presented. By considering an equations by the following form

$$x^2y'' + x(a+2bx^p)y' + (c+dx^{2q}+b(a+p-1)x^p + b2x2py=0 \quad (25)$$

which $(1-a)^2 \geq$ 4c and d, p, q are not zero. The total solution of the equation above is

$$Y = x^{\alpha}e^{-\beta x^p}\{c_1J_\mu(\lambda x^q) + c_2Y_\mu(\lambda x^q)\}$$

(26)

In this answer $\alpha = \frac{1-a}{2}$, $\beta = \frac{b}{p}$, $\lambda = \frac{\sqrt{|d|}}{q}$ and $\mu = \frac{\sqrt{(1-a)^2-4c}}{2q}$. If assume that a and b be zero from the equation (25) it is resulted

$$x^2y'' + (c + dx^{2q})\,y=0 \quad (27)$$

Which $c \leq 1/4$ and d, p, q are not zero. By dividing the both side of the equation (27) to x^2equation (28) will be resulted

$$y'' + (cx^{-2} + dx^{2(q-1)})\,y=0 \quad (28)$$

By putting equation (20) and (28) equivalent and supposing X= x-x_0, it can be written

$$Cx^{-2} + dx^{2(q-1)} \triangleq \beta^2(X) \quad (29)$$

The above equation shows that for putting both side of equation equivalent, C must be zero, d = β^2, and q = $\frac{3}{2}$, by these value $\alpha = \frac{1}{2}$, $\beta = 0$, $\lambda = \frac{2\beta}{3}$, $\mu = \frac{1}{3}$ which the relation of them is expressed above. So by these values the solution of equation will be

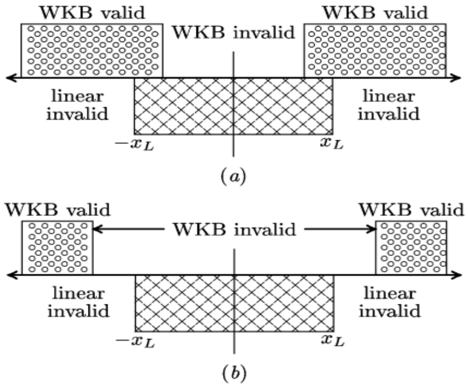

Figure.3 Validity conditions for matching the asymptotic forms of the inner functions and the outer WKB approximants. (a) Finite overlap case, so the matching is valid. (b) No overlap case, so the matching is inaccurate.

$$y=x^{\frac{1}{2}}\left\{c_1 J_{\frac{1}{3}}\left(\frac{2\beta}{3}x^{\frac{2}{3}}\right)+\ c_2 Y_{\mu}\left(\frac{2\beta}{3}x^{\frac{2}{3}}\right)\right\} \qquad (30)$$

This answer is plotted in figure .4 and it is observed that reaching to the peak which occurs at cutoff, is more monotonically. To compare solutions to each other different answer is plotted in figure.5 with each other. Of course the values of parameters in formulas are estimated and in this figure the peaks of each solution is not the same.

V. Conclusion

The numerical result of presented research has focused to attain with the well known WKB analytical method. The result which presented in this research shows that the Bessel Function Method (BFM) has reasonable agreement with the standard analytical solutions like AFM. This investigation was an attempt to explore the complete behaviors of wave in inhomogeneous plasma media. Although the use of mathematical method [7-10], Bessel function method, could not match completely to others method, WKB and AFM, but its behavior near cutoff point had good conformity. There are some discrepancies between two methods, but in many ranges they can attain to common point of view. In this paper the matter has been examined for one dimension and for future investigations it can be generalized for three dimensions. There are some estimation in this article to reduce the difficulty of explanation and computation, for example in part 3. We neglect higher order of K(x), Because the condition is for one dimension. these simplifications does not affect the main results.

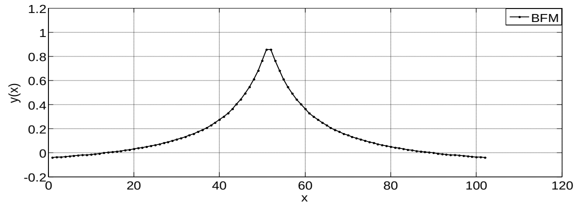

Figure.4 y(x) which is E_z(x) versus x

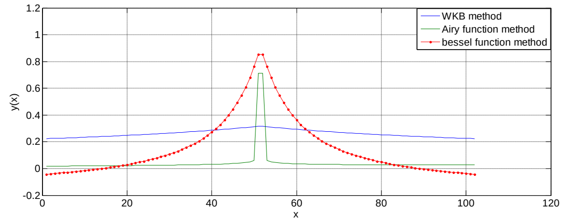

Figure.5 comparison among different solution

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