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Con S – k – Ep Matrices and Its Weighted Generalized Inverse

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ABSTRACT: *If A is a con s –k – EP matrix, then the weighted generalized inverse of A (with respect to the given matrices M,N) is a matrix which satisfies* $AA^{\dagger}A = A$, $A^{\dagger}AA^{\dagger} = A^{\dagger}$ *and that* $MAA^{\dagger}$ *and that* $AA^{\dagger}N$ *are symmetric under certain conditions on M,N.*

It is shown that the weighted generalized inverse exists if and only if $ANA^{T}MA = A$, *in which case the inverse is* $N^{T}A^{T}M^{T}$ *. When M.N are identity matrices, this reduces to the well known result that the weighted generalized inverse of a con-s-k-EP matrix when it exists, must be* A^{T} .

Keywords: *con-s-k-EP matrix, generalized inverse, weighted generalized inverse.*

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I. Introduction

Let C_{nxn} be the space of nxn complex matrices of order n. let C_n be the space of all complex n tuples. For $A \in C_{nxn}$
Let $\bar{A}$, A^T, A^*, A^S , $\bar{A}^S$, $A^{\dagger}$, R(A), N(A) and $\rho(A)$ denote the conjugate, transpose, conjugate transpose, secondary transpose , conjugate secondary transpose, Moore Penrose inverse range space, null space and rank of A respectively. A solution X of the equation AXA = A is called generalized inverse of A and is denoted by A^{-} . If $A \in C_{nxn}$ then the unique solution of the equations A XA =A , XAX = X $[AX]^* = AX$, $[XA]^* = XA[2]$ is called the Moore-Penrose inverse of A and is denoted by $A^{\dagger}$. A matrix A is called con s –$k - EP_r$ if $\rho(A) = r$ and N(A) = N(A^TVK) (or) R(A)=R(KVA^T). Throughout this paper let "k" be the fixed product of disjoint transposition in S_n= { 1,2,….n} and k be the associated permutation matrix .

Let us define the function $k(x) = (x_{k(1)}, x_{k(2)}, \ldots, x_{k(n)})$

A matrix $A = (a_{ij}) \in C_{nxn}$ is s - k - symmetric if

$a_{ij} = a_{n-k(j)+1, n-k(i)+1}$ for i.j = 1,2,…..n . A matrix $A \in C_{nxn}$ is said to be Con –s-k- EP if it satisfies the condition

$A_x = 0 <=> A^s\, k(x) = 0$ or equivalently N(A) =N(A^TVK). In addition to that A is con-s-k-EP $<=> KVA$ is con-EP or AVK is con-EP and A is con-s-k-EP$<=> A^T$ is con-s-k-EP$_r$ moreover A is said to be con-s-k-EP$_r$ if A is con-s-k-EP and of rank r. For further properties of con-s-k-EP matrices one may refer [1].

Definition 1.1

Let A, M, N $\in C_{nxn}$. The weighted Moore – Penrose inverse of A (with respect to M, N denoted by$A^{\dagger}_{M,N}$) is defined to be nxn matrix $A^{\dagger}$ satisfying

(i) $AA^{\dagger}A = A$

(ii) $A^{\dagger}AA^{\dagger} = A^{\dagger}$

(iii) $(MAA^{\dagger})^{T} = MAA^{\dagger}$

(iv) $(A^{\dagger}AN)^{T} = A^{\dagger}AN$

In case M,N are identity matrices then the matrix $A^{\dagger}$ satisfying (i) – (iv) is simply the Moore Penrose inverse (denoted by $A^{\dagger}$) of A

Theorem 1.2
Let A be an nxn con-s-k-EP matrix. Then the following assertions are equivalent:

(i) The Moore - Penrose inverse of A exists

(ii) The Moore - Penrose inverse of A exists and equals A^T.

(iii) $AA^TA = A$

(iv) $AA^TA \leq A$

(v) Any two rows of A are either identical or disjoint (ie, there is no column with a in both the rows)

(vi) Any two columns of are either identical or disjoint.

(vii) The number of ones in any 2x2 sub matrix of A is not 3.

(viii) Any 2x2 submatrix of A admits a Moore Penrose inverse.

(ix) There exist permutation matrix P.Q such that

$$PAQ = \begin{bmatrix} J_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & J_t & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Where $J_1 \dots\dots J_t$ are matrices (not necessarily square) of all ones.

(x) There exists permutation matrices P, Q Such that

$$PAQ = \begin{bmatrix} I & C \\ D & DC \end{bmatrix}$$

Where C, D satisfy $CC^T \leq I$, $D^T D \leq I$.

(xi) The main purpose of the present paper is to generalize some aspects of theorem 1.2 to the weighted case. The proof techniques are new and may be used to obtain results for matrices over more general structures. Thus most of our statements are valid for matrices over a distributive lattice. Whereas some require the structure of a completely ordered set. Such generalizations will be clear from the proofs. However, we have chosen to present the results only in the setting of con-s-k-EP matrices. In the next section we consider the question of the existence of a weighted Moore – Penrose inverse and give a formula for it when it exists

II. The Main Result

we begin by showing that under some conditions on M,N the inverse $A^{\dagger}_{M,N}$. when it exits is unique , we denote the row space of the matrix A by R(A), and the column space by R(A) .

THEOREM 2.1
Let A be a con-s-k-EP matrix of order nxn and M,N be matrices order nxn. R (KVA^T)= R(MA), R(KVA^T) = R(AN). There exist matrices x,y suchthat XMA =A, ANY =A. Then

(a) AN^TA^T = A NA^T, A^TM^T A = A^TMA

(b) $A^{\dagger}$ M,N is unique.

Proof :

(a) Let $A^{\dagger} = A^{\dagger}_{M,N}$ exists then

A$N^T A^T$ = A$N^T A^T (A^T)^\dagger A^T$ (since A$A^\dagger$A = $A^\dagger$)

= A$A^\dagger$AN A^T (since $A^\dagger AN$ $is\ symmetric)$

= AN A^T(since A$A^\dagger$A = A)

The proof of the remaining part of
(a) Is similar to the above

(b) Let , if possible $A_1^\dagger$,$A_2^\dagger$ be two candidates for $A_{M,N}^\dagger$ then

$A_1^\dagger$AN = $A_1^\dagger$ $AA_2^\dagger$ AN (since A = A$A_2^\dagger A)$

= $A_1^\dagger$AN$A^T A_2^{\dagger T}$ (since = $A_2^\dagger$AN is symmetric)

= $A_1^\dagger$A$N^T A^T A_2^{\dagger T}$ (using (a))

= $N^T A^T A_1^{\dagger T} A^T A_2^{\dagger T}$ (since $A_1^\dagger$AN is symmetric)

= $N^T A^T A_2^{\dagger T}$ (since A$A_1^\dagger A = A)$

= $A_2^\dagger$AN (since $A_2^\dagger$AN is symmetric).

Thus $A_1^\dagger$ANY = $A_2^\dagger$AN Y and hence $A_1^\dagger$A = $A_2^\dagger$A (since ANY =A) . it follows

$A_1^\dagger$ A $A_1^\dagger$ = $A_2^\dagger$A$A_1^\dagger$ and therefore

$A_1^\dagger$ = $A_2^\dagger A$ $A_1^\dagger$ (1)

Now

MA$A_1^\dagger$ = M A $A_2^\dagger$ A $A_1^\dagger$ (since A = A$A_2^\dagger$A)

= $A_2^{\dagger T}$ $A^T M^T AA_1^\dagger$ (since MA$A_2^\dagger$ is symmetric)

= $A_2^{\dagger T}$ $A^T MAA_1^\dagger$ (using (a))

= $A_2^{\dagger T}$ A^T $A_1^{\dagger T}$ $A^T M^T$ (since MA$A_1^{\dagger T}$ is symmetric)

= $A_2^{\dagger T}$ $A^T M^T$ (since A$A_1^\dagger$A =A)

= M A $A_2^\dagger$ (since MA$A_2^\dagger$ is symmetric)

It follows that X M A $A_1^\dagger$ = X M A $A_2^\dagger$ and hence

A $A_1^\dagger$ = A $A_2^\dagger$ (since XMA =A)

Therefore $A_2^\dagger$A $A_1^\dagger$ = $A_2^\dagger AA_2^\dagger$ and thus

$A_2^\dagger$A $A_1^\dagger$ = $A_2^\dagger$

It follows from (1) (2) that $A_1^\dagger$ = $A_2^\dagger$ and the proof is complete.

Example

A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $A_1^\dagger$ = $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

Take M=I and N to be the 2x2 zero matrix. Then it can be verified that both $A_1^\dagger$, $A_2^\dagger$ satisfy all conditions in definition 1 and therefore the weighted generalized inverse is not unique in this example observe that here the condition of theorem 2.1 is not satisfied
The next result will be used in the sequence

Lemma 1:

Let A be an nxn matrix . then A≤ KVA$A^T AVK$

Proof :

Let B = KVAA^TAVK we must show that $a_{ij} \leq b_{ij}$ for all i, j this is b_{ij} = KV [$\sum_{k=1}^{n} \sum_{i=j}^{m} a_{ik}\, a_{lk}\, a_{lj}$]
VK (3)

If we set k = j , ℓ=I then $a_{ik}\, a_{lj}\, a_{li}$ = a_{ij}^3 = a_{ij}. It follows from (3) that b_{ij}=1 and the proof complete .

The following is the main result of this section.

Theorem 2.2

(a) R(KVA^T) = R(MA), R(KVA^T) = R(AN)

(b) M $\geq$ I , N $\geq$ I

Then the following assertations are equivalent:

(i) $A^{\dagger}_{M,N}$ exists

(ii) Any one of the following holds

(1) ANA^TMA = A (2) A$N^T A^T$MA = A

(3) AN$A^T M^T$ A = A (4) A$N^T A^T M^T$ A =A

And thus $A^{\dagger}_{M,N}$= $N^T A^T M^T$

(iii) Any two rows of A are either identical or disjoint, and ANA^T=AA^T

(iv) Any two columns of A are either identical or disjoint and ANA^T=AA^T

A^TMA = A^TA.

Proof:

(i) $\Rightarrow (ii)$

Suppose $A^{\dagger}$ = $A^{\dagger}_{M,N}$ exists since the number of con –s – k - EP matrices of a given order is finite there exists integers k$\geq$ I , s$\geq$ i such that

$(ANA^T M^T)^K$ = $(ANA^T M^T)^{K+S}$ (4)

Without loss we can assume s $>$ 1 for if s=1 then (4) clearly holds for S=2 as well as now left multiplying equation (4) by $A^{\dagger}$ and then using the fact $A^{\dagger}$AN is symmetric, we get

$$N^T A^T A^{\dagger T} M^T (ANA^T M^T)^{K-1} = N^T A^T A^{\dagger T} M^T (ANA^T M^T)^{K-1+S}$$

Left multiply the above equation by Y^T and then use AGA =A, ANY=A to get $A^T M^T (ANA^T M^T)^{K-1}$= $A^T M^T (ANA^T M^T)^{K-1+S}$.

Left multiply the above equation by $A^{\dagger T}$ and then use the facts that MA$A^{\dagger}$ is symmetric and the A$A^{\dagger} A = A$ to get

M$(ANA^T M^T)^{k-1}$ = M$(ANA^T M^T)^{K-1+S}$. Finally left multiply the above equation by x and use XMA=A to get

$(ANA^T M^T)^{K-1}$ = $(ANA^T M^T)^{K-1+S}$. Continuing this way we may assume k=1, without less of generality and therefore

AN$A^T M^T$ = $(ANA^T M^T)^{S+1}$

Starting with the above equation, we get the following chain of implications:

$\Rightarrow A^{\dagger} ANA^T M^T = A^{\dagger}ANA^T M^T (ANA^T M)^S$

$\Rightarrow N^T A^T A^{\dagger T} A^T M^T = N^T A^T A^{\dagger T} A^T M^T (ANA^T M^T)^S$

$\Rightarrow N^T A^T M^T = N^T A^T M^T (ANA^T M^T)^S$

$\Rightarrow Y^T N^T A^T M^T = Y^T N^T A^T M^T (ANA^T M^T)^S$

$\Rightarrow A^T M^T = A^T M^T (ANA^T M^T)^S$

$\Rightarrow A^T M^T X^T = A^T M^T (ANA^T M^T)^S X^T$

$\Rightarrow A^T = A^T M^T (ANA^T M^T)^{S-1} ANA^T$

And therefore

A= $(ANA^T M)A$ (5)

By Lemma 1.

A$\leq$ A $A^T A$, and so since M$\geq$ I, N$\geq$I we have

A$\leq$ A$N^T A^T$MA (6)

And hence post multiplying by $N^T A^T$MA we get

A$N^T A^T$MA $\leq (AN^T A^T M)^2$ A (7)

Repeated post multiplication of (7) by $N^T A^T$MA gives

A$\leq$ A$N^T A^T$MA $\leq$ $(AN^T A^T M)^2$ A

$\leq (AN^T A^T M)^3 A \leq$ ……..$\leq (AN^T A^T M)^2$ A = A

Where the equality follows in view of (5)

$A = AN^T A^T MA = ANA^T MA = ANA^T M^T A = AN^T A^T M^T A,$ (8)

Where the last three equalities follow in view of Theorem 2.1(a) now we show $A^\dagger = A^\dagger_{M.N} = N^T A^T M^T$ We have shown

$AA^\dagger A = A$, but $A^\dagger AA^\dagger = N^T A^T M^T AN^T A^T M^T = N^T A^T M^T$ (in view of (8)) and

MAG = MA$N^T A^T M^T$

= MANAM^T (Using Theorem 2.1 (a))

= $(MAN^T A^T M^T)^T$ = $(MAA^\dagger)^T$

So $MAA^\dagger$ is symmetric showing $A^\dagger AN$. symmetric is similar .

so $A^\dagger_{M,N} = N^T A^T M^T$

(ii) $\Rightarrow$ (i) let ANA^TMA =A. By Lemma 1

A$\leq$ A$A^T A$ as M $\geq$ I , N$\geq$ I we have

A$\leq$A$A^T A$ $\leq$ ANA^TA$\leq$ ANA^TMA =A thus

= A$A^T A$ =A = ANA^TA (9)

The second part of the above equation gives

AA^T $= ANA^T$AA^T

= ANA^T (using the first part of (9))

= A$N^T A^T$ (since AA^T is symmetric)

Similarly it can be shown that A^TMA = $A^T M^T$A = $A^T A$

Now using these facts equation (9) and the assumption one can easily see that $A^\dagger_{M,N} = N^T A^T M^T$.

(ii) $\Rightarrow$ (iii) without loss we take ANA^TMA =A

Suppose two rows of A, say the i th and jth are not disjoint. Then there exists k such that $a_{ik} = a_{jk}$ =1 . now if a_{ir} =1 for some r, then we have

$a_{jr} \geq a_{jk} n_{kk} a_{ik} M_{ir} a_{ir}$ = 1

and hence a_{jr} =1 . thus the ith row of A is entry wise dominated by the jth row. Similarly we can shown that the j th row of A is entry wise dominated by the ith row and hence the two rows must be identical.

The proof of the remaining part is essentially contained in the proof of (ii) $\Rightarrow$ (i)

(iii)$\Rightarrow$ (ii) : let B = A$A^T A$ and suppose b_{ij}=1. So there exists $l_1 l_2$ such that

$a_i b_1$ = a$l_2 t_1$ =al_2j =1

Now observe that the l_1 the column of A is nonzero . so by hypothesis we have that the ith row of A is equal to the l_2 the row of A. but we also have $a_{l_2} j = 1.$

So , a_{ij}=1 and therefore A$A^T A \leq$ A. it follows by lemma 1 that A = AA^TA.

It follows by lemma 1 that A = A$A^T A$. Since ANA^T= AA^T, A^TMA = A^TA.

Then ANA^TMA =A$A^T MA$ =A$A^T A$= A

and (ii) is proved. The equivalence of (iv) and (ii) is proved similarly that completes the proof of the theorem.

An examination of the proof of theorem 2.2 reveals that condition (b) may be replaced by the weaker condition ANA^TMA$\geq$ A.

We now provide a proof of theorem 1.1

Proof of the 1.1 . the equivalence of (i) – (iv) of the theorem essentially follows from theorem 2.2 by setting M=N=I. the equivalence of (v) and (vii) is easy to prove and so is the equivalence of (vii) and (viii). The implication (v)$\Rightarrow$ (ix) and (ix) $\Rightarrow$ (iii) are easy to prove thus we have shown that assertions (i) –(ix) are equivalent.

It is easy to see that assertions (i) –(ix) are equivalent.

It is easy to see that (ix) $\Rightarrow$ (x). if (x) holds then it can be verified that A^T is the Moore Penrose inverses of A and this (i) holds that completes the proof

We remark that all the assertions in theorem 1.1 except (vii) .(viii) are essentially contained in the literatures. See [2,4,5] however, we have given proofs for completes the proof.

As shown in theorem 1.1 if A admits a Moore Penrose inverse, then it must be A^T some times it happens that the weighted Moore Penrose inverse $A^{\dagger}_{M.N}$ = A^T, the trivial case being M = N = I. So the obvious question is whether we can precisely point out the cases when $A^{\dagger}_{M.N}$ = A^T. To answer this question we need the following result.

Theorem 2.3:

Let A,M,N be as in Theorem 2.2 then the following are equivalent

(i) $A^{\dagger}_{M,N}$ exists

(ii) There exists permutation matrices P and A such that

$$\bar{A} = PAQ = \begin{pmatrix} j_1 & 0 & \dots & 0 & 0 \\ 0 & J_2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & J_K & 0 \\ 0 & 0 & & 0 & 0 \end{pmatrix}$$

$$\bar{N} = \begin{pmatrix} N_{ij} & \dots & N_{I1,K+1} \\ \vdots & & \vdots \\ N_{K+1} & \dots & N_{K+1,K+1} \end{pmatrix}$$

A conformed partition and then using (10) we see that all the blocks N_{ij}= 0

I$\leq$ i, j $\leq$ k , i$\neq$j and that $N_{ij} \neq$ 0, I$\leq$ i$\leq$k. We have a similar conclusion regarding $\bar{M}$.

It is easy to see that $\bar{A}^{\dagger} = \bar{N}^T \bar{A}^T \bar{M}^T$ is the weighted Moore – Penrose inverse of $\bar{A}$ with respect to $\bar{M}$,$\bar{N}$. But

$\bar{A}^{\dagger}$= $(Q^T N^T Q)$ $(Q^T A^T P^T)$ (P$M^T P^T$) = $Q^T (N^T A^T M^T) P^T$

= $Q^T A^{\dagger} P^T$

Now carrying out the block multiplication in the equation $\bar{A}^{\dagger}$= $\bar{N}^T \bar{A}^T \bar{M}^T$. We See that $\bar{A}^{\dagger}$ is of the form gives in the statement

Now by (ii), the proof of (i) $\Rightarrow$ (ii) is complete.

Conversely, suppose (ii) holds defining $\bar{A}^{\dagger}$ as in the statement of the theorem, it is easy to check that $\bar{A}\,\bar{A}^{\dagger}\bar{A} = \bar{A}$.
Since $\bar{A}^{\dagger} = \bar{N}^{T}\bar{A}^{T}\bar{M}^{T}$.
We have $\bar{A}\,\bar{N}^{T}\bar{A}^{T}\bar{M}^{T}\bar{A} = \bar{A}$.

This implies PA$N^{T}A^{T}M^{T}$AQ = PAQ.

Therefore A$N^{T}A^{T}M^{T}$A = A and thus $A^{\dagger}_{M,N}$ by theroem 2.2.
As a simple corollary we state the following result without proof.

Corollary 1:

Let A,M,N be as in theorem 2.2 then $A^{\dagger}_{M,N} = A^{T}$ if and only if condition (ii) of theorem 2.3 is satisfied with the additional proviso that $\bar{M}$ and $\bar{N}$ are block diagonal .
We also have the following

Corollary 2:

Let A,M,N be as in Theorem 2.2 and further suppose A has no zero row or zero column then if $A^{\dagger}_{M,N}$ exists it equals A^{T}.

Proof :

Observe that $\bar{A}$ has no diagonal zero block. Hence by Theorem 2.3 $\bar{M}, \bar{N}$ are block diagonal furthermore that $A^{\dagger}_{M,N}$

Exists implics that $\bar{A}^{\dagger}$ exists . the result now follows by corollary 1.we conclude with an example which shows that the condition that A has no zero row or column is necessary in corollary 2.

Example:

Let A = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ N = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ M = I

Then ANA^{T}MA = A but N$_{A^{T}M}$ = $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \neq A^{T}$

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