International Journal of Modern Engineering Research (IJMER)
www.ijmer.com Vol.2, Issue.6, Nov-Dec. 2012 pp-4195-4200 ISSN: 2249-6645

Existence of Hopf-Bifurcations on the Nonlinear FKN Model

Tarini Kumar Dutta, [1] and Nabajyoti Das, [2]

1. Professor, Department of Mathematics, Gauhati University, Guwahati -781014 : India
2. Assistant Professor, Department of Mathematics, Jawaharlal Nehru College, Boko-781123: India

Abstract: *The principal objectives of this paper are (i) to study the development of a general theory for evaluating supercritical and subcritical Hopf bifurcation in any nonlinear differential equations, and (ii) to determine supercritical and subcritical Hopf bifurcations in a rigorous manner on the the Field-Körös-Noyen or FKN model:*

$$\frac{dx}{dt} = \varepsilon^{-1}(qy - xy + x - x^2), \frac{dy}{dt} = \delta^{-1}(-qy - xy + 2fz), \frac{dz}{dt} = x - z.$$

Where $\varepsilon, \delta, q, f$ are adjustable parameters?

Key Words: *Supercritical Hopf bifurcation, Subcritical Hopf bifurcation, nonlinear differential equation*
2010 AMS Classification: 37 G 15, 37 G 35, 37 C 45

I. Introduction

There has been considerable interest recently in sustained oscillation in chemically reacting systems represented by a set of nonlinear differential equations. These oscillations can be periodic in which case the concentrations of some species undergo regular variations with time or they can be non-periodic in which case the reactor never approaches a globally attracting limit cycle. This later condition has been termed chemical chaos.

Many of these studies have been carried out with the well known Belouson-Zhabotinskii reaction. Periodic chemical reaction such as the Belousov-Zhabotinski reaction provide wonderful example of relaxation oscillation in science [3, 9]. The BZ reaction is one of the first oscillating reactions which is studied systematically [1]. Although there are many reactions involved in the BZ reaction they can be rationally reduced to 5 key reactions, with known values for the rate constants, which capture the basic elements of the mechanism. These five reactions can then be represented by a 3-chemical system in which the overall rate constants can be assigned with reasonable confidence. The model is known as the **Field-Körös-Noyen or FKN model** [6]**:**

$$\frac{dx}{dt} = \varepsilon^{-1}(qy - xy + x - x^2),$$
$$\frac{dy}{dt} = \delta^{-1}(-qy - xy + 2fz),$$
$$\frac{dz}{dt} = x - z.$$

We first highlight some related concepts for completeness of our exploration.

II. Limit cycles

A cyclic or periodic solution of a nonlinear dynamical system corresponds to a closed loop trajectories in the state space. A trajectory point on one of these loops continues to cycle around that loop for all time. These loops are called **cycles**, and if trajectories in the neighborhood to the cycle are attracted toward it, we call the cycle a **limit cycle**. Some limit cycles are shown in the figure 1, where (a) shows an inner limit cycle, (b) an outer limit cycle, (c) a stable limit cycles, (d) an unstable limit cycle, and (e) and (f) periodic orbit that may be called saddle limit cycles [4, 7].

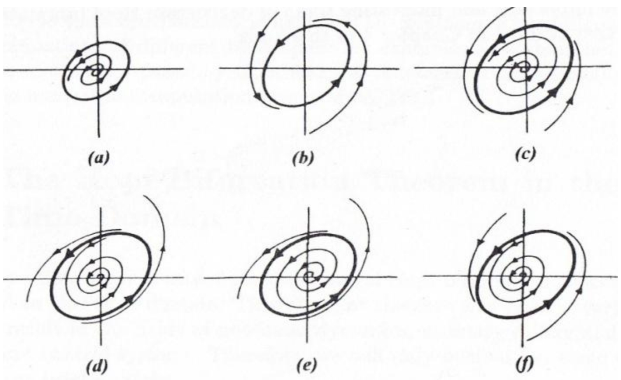

Fig1. Periodic orbits and limit cycles.

III. The Hopf bifurcation theorem in continuous-time

In this discussion we will restrict our discussion on second-order systems of nonlinear ordinary differential equations, although almost all the results and discussions given below can be extended to general nth-order systems. We consider the system:

$$\frac{d\overline{\mathbf{x}}}{dt} = \xi(\overline{\mathbf{x}};b), \quad \overline{\mathbf{x}} \in \Re^2 \tag{1.1}$$

where b denotes a real parameter on an interval *I*. We assume that the system is well defined, with certain smoothness on the nonlinear vector field ξ, and has a unique solution for each given initial value $\overline{\mathbf{x}}(t_0) = \overline{\mathbf{x}}$ for each fixed $b \in I$. We also assume that the system has an equilibrium point $\underline{\mathbf{x}}^*(b)$ and that the associated Jacobian $J = \left.\frac{\partial \xi}{\partial \overline{\mathbf{x}}}\right|_{\overline{\mathbf{x}}=\overline{\mathbf{x}}^*}$ has a single pair of complex conjugate eigenvalues $\eta(\mathrm{b}), \overline{\eta(\mathrm{b})} = \mathrm{Re}\,\eta \pm \mathrm{Im}\,\eta.$ Now suppose that this pair of eigenvalues has the largest real part of all the eigenvalues and is such that in a small neighborhood of a bifurcation value b_c, (i) $\mathrm{Re}\,\eta < 0$ if $b < b_c$, (ii) $\mathrm{Re}\,\eta = 0, \mathrm{Im}\,\eta \neq 0$ if $b = b_c$ and (iii) $\mathrm{Re}\,\eta > 0$ if $b > b_c$. Then, in a small neighborhood of $b_c, b > b_c$, the steady state is unstable by growing oscillations and, at least, a small amplitude limit cycle periodic solution exists about the equilibrium point. The appearance of periodic solutions (all depend on the particular nonlinear function ξ) out of an equilibrium state is called Hopf bifurcation. When the parameter b is continuously varied near the criticality b_c, periodic solutions may emerge for $b < b_c$ (this case is referred to as **supercritical bifurcation**) or for $b > b_c$ (which is referred to as **subcritical bifurcation**) [2, 6, 8].

Armed with these concepts, we now concentrate to our main study and investigation.

IV. The principal investigation

We consider a two-dimensional system $\dot{\overline{\mathbf{x}}} = \xi(\overline{\mathbf{x}};b),\ b \in \Re,\ \overline{\mathbf{x}} = (x, y) \in \Re^2$ where ξ depends smoothly on the real variable parameter b such that for each b near the origin $(0,0)$ there is an equilibrium point $\overline{\mathbf{x}}^*(b)$ with the Jacobian matrix $D\xi_{\underline{\mathbf{x}}}(\overline{\mathbf{x}}^*(b), b)$ having a complex conjugate pair of eigenvalues $\eta(b), \overline{\eta}(b) = \varphi \pm i\psi$ which cross the imaginary axis as the parameter b passes through $(0,0)$. Using complex coordinate $z = x + iy$, the system can be expressed in the variable z as

$$\dot{z} = \eta z + A_1 z^2 + B_1 z\bar{z} + C_1 \bar{z}^2 + M_1 z^2 \bar{z} + \ldots \tag{1.2}$$

where A_1, B_1, C_1, M_1 are complex constants. By making a suitable change of variables the system can be transformed to a normal form:

$$\dot{w} = w(\eta + a|w|^2) + o(|w|^4), \tag{1.3}$$

where w, a are both complex numbers. We write $a = k + il;\ k, l \in \Re.$ The behavior of the system (1.3) is most conveniently studied using polar coordinate $w = re^{i\theta}$. From this we obtain, $\dot{w} = e^{i\theta}\dot{r} + ire^{i\theta}\dot{\theta}.$ Hence $\dot{r} = r^{-1}\,\mathrm{Re}(\overline{w}\dot{w})$ and $\dot{\theta} = r^{-2}\,\mathrm{Im}(\overline{w}\dot{w})$ and then (1.3) implies

$$\dot{r} = kr^3 + o(r^4),\ \dot{\theta} = \psi + o(r^2) \tag{1.4}$$

Supercritical and subcritical Hopf bifurcation occur according as $k < 0$ and $k > 0$ respectively. If $k = 0$, considering high order terms we can draw the same conclusion [2].

V. Determination of the indicator of bifurcations: *k*

Here we are interested in finding the expression for *k*, whose sign determines the supercritical and subcritical Hopf bifurcation. For this we need the term in $z^2\bar{z}$. In order to eliminate the quadratic terms, we apply the transformation $w = z + \delta z^2 + \rho z\bar{z} + \theta\bar{z}^2$. Then we expand $\dot{w}$, keeping only terms upto second order (and noting, for example that the difference between z^2 and w^2 is third order, so z^2 can be replaced by w^2 etc.). We have

$$\begin{aligned}
\dot{w} &= \dot{z} + 2\delta\, z\dot{z} + \rho\, \bar{z}\dot{z} + \rho z\dot{\bar{z}} + 2\theta\, \bar{z}\, \dot{\bar{z}} \\
&= \eta z + A_1 z^2 + B_1 z\bar{z} + C_1 \bar{z}^2 + M_1 z^2 \bar{z} + 2\delta z(\eta z + B_1 z\bar{z}) + \\
&\quad \rho\, \bar{z}(\eta z + A_1 z^2) + \rho\, z(\overline{\eta}\bar{z} + \overline{B}_1 z\bar{z}) + 2\theta\, \bar{z}(\overline{\eta}\bar{z} + \overline{C}_1 z^2)
\end{aligned}$$

where cubic terms are neglected other than $z^2\bar{z}$. We eliminate the quadratic terms by putting

$$\delta = -A_1/\eta = iA_1/\psi,\ \rho = \text{-i}B_1/\psi,\ \theta = -iC_1/3\psi.$$

Then we obtain

$$\dot{w} = \eta w + \left(M_1 + iA_1B_1/\psi - i|B_1|^2/\psi - 2i|C_1|^2/3\psi\right)w^2\bar{w},$$

where again cubic terms are neglected other than $w^2\bar{w},$ and terms of order higher than 3. We conclude that

$a = M_1 + iA_1B_1/\psi - i|B_1|^2/\psi - 2i|C_1|^2/3\psi$.

and

$$\begin{aligned} k &= \mathrm{Re}\left(M_1 + iA_1B_1/\psi\right) \\ &= \mathrm{Re}(M_1) - \psi^{-1}\,\mathrm{Im}(A_1B_1). \end{aligned}$$

VI. Extension to three order differential equations

Let us assume that we have a three-dimensional system:

$$\dot{\bar{\mathbf{x}}} = \xi(\bar{\underline{\mathbf{x}}}),\quad \bar{\mathbf{x}} = (x,y,z)^T, (x,y,z) \in \Re^3$$

which has an equilibrium point for which there is one negative eigenvalue and an imaginary pair. The behavior of the system near the equilibrium point can be analyzed by a reduction of the system to a two-dimensional one, as follows. First we choose coordinates so that the equilibrium point is the origin and so that the linearised system is

$$\dot{v} = \rho v, \qquad \dot{z} = \lambda z$$

where v is a real variable and z is complex, and $\rho < 0,\ \lambda = i\sigma.$

We can now express the system as

$$\dot{v} = \rho v + \alpha vz + \bar{\alpha}v\bar{z} + \gamma z^2 + \delta z\bar{z} + \bar{\gamma}\,\bar{z}^2 + \ldots..$$

$$\dot{z} = \lambda z + pvz + qv\bar{z} + rz^2 + sz\bar{z} + t\,\bar{z}^2 + dz^2\bar{z} + \ldots..$$

If the equation for *v* were of the form $\dot{v} = \rho v + vf(v,z)$ then the plane *v = 0* would be invariant, in the sense that solutions starting on this plane stay on it, and we could restrict attention to the behavior on this plane. What we do below is to find a change of variables which converts the system into one which is sufficiently close to this form. We try the change of variables

$$v = w + az^2 + bz\bar{z} + \bar{a}\bar{z}^2, \text{where } b \text{ is real.}$$

We obtain

$$\dot{w} = \rho w + \rho az^2 + \rho bz\bar{z} + \rho\bar{a}\bar{z}^2 + \alpha wz + \bar{\alpha}w\bar{z} + \gamma\, z^2 + \delta z\bar{z} + \bar{\gamma}\,\bar{z}^2 - 2a\lambda z^2 - 2\bar{a}\bar{\lambda}\bar{z}^2,$$

neglecting terms of order 3 and higher. Then if we choose

$$a = \gamma \div (2i\sigma - \rho)$$

and $$b = -\delta \div \rho$$

We have

$$\dot{w} = \rho w + \alpha wz + \bar{\alpha}w\bar{z} + \ldots$$

which is of the desired form (as far as of second-order, which turns out to be sufficient). Putting $w = 0$, in the equation for $\dot{z},$ and retaining only terms of order second and those involving $z^2\bar{z},$ we obtain

$$\dot{z} = \lambda z + rz^2 + sz\bar{z} + t\bar{z}^2 + \left(\frac{-p\delta}{\rho} + \frac{q\gamma}{2i\sigma - \rho} + d\right)z^2\bar{z}$$

and using the two-dimensional theory we obtain

$$k = \text{Real part of}\left(\frac{-p\delta}{\rho} + \frac{q\gamma}{2i\sigma - \rho} + d + \frac{irs}{\sigma}\right).$$

Supercritical and subcritical Hopf bifurcation occur according as $k < 0$ and $k > 0$ respectively. If $k = 0,$ considering high order terms we can draw the same conclusions.

VII. Our main study

For our main study we consider the Field-Körös-Noyen or FKN model:

$$\frac{dx}{dt} = \varepsilon^{-1}(qy - xy + x - x^2),$$
$$\frac{dy}{dt} = \delta^{-1}(-qy - xy + 2fz), \tag{1.5}$$
$$\frac{dz}{dt} = x - z.$$

For our purpose, the parameters are fixed as in the FKN model as given below [6]:

$$\varepsilon = 5\times10^{-5} = 0.00005, \quad \delta = 2\times10^{-4} = 0.0002, \quad q = 8\times10^{-4} = 0.0008, \quad f = 0.5$$

With these parameter values the equilibrium points (x^*, y^*, z^*) of the system (1.5) are given by setting the left-hand sides zero and solving the resulting system of equations, to get

$$(x^* = 0,\ y^* = 0,\ z^* = 0),$$
$$or\ (x^* = -0.0404019999500025,\ y^* = 1.0202009999750015,\ z^* = -0.0404019999500025),$$
$$or\ (x^* = 0.0396019999500025,\ y^* = 0.9801990000249986,\ z^* = (0.0396019999500025).$$

These numerical solutions are found with the help of MATHEMATICA. Out of these equilibrium points

$$(x^* = -0.0404019999500025,\ y^* = 1.0202009999750015,\ z^* = -0.0404019999500025)$$

is suitable for our purpose.

Let us take a linear transformation which moves the equilibrium point to the origin. We take $u = x - x^*$, $\mathrm{v} = \mathrm{y} - y^*$ and $w = \mathrm{z} - z^*$.

Then the system (1.5) becomes

$$\frac{du}{dt} = \varepsilon^{-1}(q(v + y^*) - (u + x^*)(v + y^*) + (u + x^*) - (u + x^*)^2) \tag{1.6}$$
$$= 1.38778\times10^{-13} - 20000u^2 + u(1212.06 - 20000v) + 824.04v$$

$$\frac{dv}{dt} = \delta^{-1}(-q(v + y^*) - (u + x^*)(v + y^*) + 2f(w + z^*)) \tag{1.7}$$
$$= 3.46945\times10^{-14} + u(-5101 - 5000v) + 198.01v + 5000w$$

$$\frac{dw}{dt} = (u + x^*) - (w + z^*) \tag{1.8}$$
$$= 0 + x - z$$

The matrix of linearized system is then of the form

$$M = \begin{bmatrix} 1212.0599985000702 & 824.0399990000501 & 0.0 \\ -5101.004999875007 & 198.0099997500125 & 5000.0 \\ 1.0 & 0.0 & -1.0 \end{bmatrix}$$

The eigenvalues $\rho, \lambda_1, \lambda_2$ of M are

$$\rho = -0.07276521654210118,$$
$$\lambda_1 = 704.5713817333121 + 1986.3795683652036i,$$
$$\lambda_2 = 704.5713817333121 - 1986.3795683652036i$$

Let us take $$D = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

as the diagonal matrix. Then we obtain

$$M^{-1}D = \begin{bmatrix} -0.0000445755166 & -1.7962166905241-5.0640264803776i & -8981.0834526207+25320.132401888i \\ -0.0000227379933 & 2.6420105842731+7.4485509631315i & 13210.052921365-37242.754815658i \\ -0.0000445755166 & -1.7962166905241-5.0640264803776i & -9685.6548343540+27306.511970253 \end{bmatrix}$$

In order to make the linearized system into a diagonal form, we make the coordinate change by $M^{-1}DU,$ where U is the column matrix, $U=[f,g,h]^T$.

$$\text{Now } M^{-1}DU = \begin{bmatrix} (-0.0000445755+0.i)f-(1.79622+5.06403.i)g-(8981.08-25320.1.i)h \\ (-0.000022738+0.i)f+(2.64201+7.44855.i)g+(12310.1-37242.8.i)h \\ (-0.0000445755+0.i)f-(1.79622+5.06403.i)g-(9685.65-27306.5.i)h \end{bmatrix}$$

Putting

$$u=(-0.0000445755+0.i)f-(1.79622+5.06403.i)g-(8981.08-25320.1.i)h,$$
$$v=(-0.000022738+0.i)f+(2.64201+7.44855.i)g+(12310.1-37242.8.i)h,$$
$$w=(-0.0000445755+0.i)f-(1.79622+5.06403.i)g-(9685.65-27306.5.i)h$$

in equations (1.6) and (1.7), we get

$$\frac{du}{dt}=1.38778\times10^{-13}-20000(-0.0000445755f-(1.79622+5.06403i)g-(8981.08-25320.1i)h)^2$$
$$+(-0.0000445755f-(1.79622+5.06403i)g-(8981.08-25320.1i)h)(1212.06-$$
$$20000(-0.0000445755f-(1.79622+5.06403i)g-(8981.08-25320.1i)h))$$
$$+824.04(-0.000022738f+(2.64201+7.44855i)g+(13210.1-37242.8i)h)$$

$$\frac{dv}{dt}=3.46945\times10^{-14}+(-0.0000445755f-(1.79622+5.06403i)g-(8981.08-25320.1i)h)(-51101-$$
$$5000(-0.000022738f+(2.64201+7.44855i)g+(13210.1-37242.8i)h))+$$
$$198.01(-0.000022738f+(2.64201+7.44855i)g+(13210.1-37242.8i)h)+$$
$$5000(-0.0000445755f-(1.79622+5.06403i)g-(9685.65-27306.5i)h).$$

Finally, under the stated transformation (as described in General theory) the system becomes

$$\frac{du}{dt}=-0.0727652f+(-1.66416-4.69172i)fg+(-8320.8+23458.6i)fh$$
$$+(-211121+171325i)g^2+(2.7189\times10^9+3.77936\times10^{-8}i)gh \tag{1.9}$$
$$+(-5.27803\times10^{12}-4.28312\times10^{12}i)h^2+...$$

$$\frac{dv}{dt}=(704.571+1986.38i)g+(0.384633+1.084339i)fg+(1923.17-5421.93i)fh$$
$$+(-164870+133792i)g^2+(2.12326\times10^9+1.93249\times10^{-8}\mathrm{i})gh$$
$$+(-4.12175\times10^{12}-3.3448\times10^{12}i)h^2+0.0g^2h+... \tag{1.10}$$

From above, we obtain

ρ = Coefficient of *f in* (1.9) = $-0.0727652,$

p = Coefficient of fg in (1.10) = $0.384633+1.08439i,$

δ = Coefficient of gh in (1.9) = $2.7189\times10^9+3.77936\times10^{-8}i,$

q = Coefficient of fh in (1.10) = $1923.17-8421.93i,$

γ = Coefficient of g^2 in (1.9) = $-211121+171325i,$

d = Coefficient of g^2h in (1.10) = $0,$

r = Coefficient of g^2 in (1.10) = $-164870+133792i,$

s = Coefficient of gh in (1.10) = $2.12326\times10^9+1.93249\times10^{-8}i,$

σ =Imaginary part of eigenvalues = 1986.37957

Using the above values we can calculate the value of k as

$$k = \text{Real part of}\left(\frac{-p\delta}{\rho} + \frac{q\gamma}{2i\sigma - \rho} + d + \frac{irs}{\sigma}\right).$$

$$\approx -1.28639 \times 10^{11}$$

Hence, we have a supercritical Hopf bifurcation. Similarly, we can study the Hopf bifurcation of a given system for different values of the parameters.

VIII. Conclusion

We think, our method is quite suitable for obtaining Hopf Bifurcation for any order nonlinear differential equations, if Hopf bifurcation exists.

References

[1] Atkins, P. and Paula, J. D., Atkin's Physical Chemistry, Seventh edition, oxford University Press, 2002
[2] Das, N. and Dutta, T. K., Determination of supercritical and subcritical Hopf bifurcation on a two-dimensional chaotic model, International Journal of Advanced Scientific Research and Technology, Issue2, Vol. 1, February, 2012
[3] Field, R. and Burger, M., Oscillations and Travelling Waves in Chemical Systems, Wiley, New York, 1985
[4] Hilborn, Robert C., Chaos and Nonlinear Dynamics, Oxford University Press, 1994
[5] Moiola, J. L. and Chen, G., Hopf Bifurcation Analysis: a frequency domain approach, World Scientific, 1996
[6] Murray, J. D., Mathematical Biology I: An Introduction, Third Edition (2002), Springer
[7] J. L Moiola and G. Chen, Hopf Bifurcation Analysis: a frequency domain approach, World Scientific, 1996
[8] Roose, D. and Hlavacek, V., A Direct Method for the computation of Hopf bifurcation points, SIAM J. APPL. MATH., Vol. 45, No. 6, December 1985
[9] Scott, S. K., Oscillations, Waves, and Chaos in Chemical Kinetics, Oxford Science Publications, Oxford, UK, 1994