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On The Zeros of Polynomials

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Abstract: In this paper we extend Enestrom -Kakeya theorem to a large class of polynomials with complex coefficients by putting less restrictions on the coefficients . Our results generalise and extend many known results in this direction.

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I. Introduction and Statement of Results

Let P(z) be a polynomial of degree n. A classical result due to Enestrom and Kakeya [9] concerning the bounds for the moduli of the zeros of polynomials having positive coefficients is often stated as in the following theorem(see [9]) :

Theorem A (Enestrom-Kakeya) : Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$0 \le a_1 \le a_2 \le ...... \le a_n .$$

Then P(z) has all its zeros in the closed unit disk $|z| \le 1$.

In the literature there exist several generalisations of this result (see [1],[3],[4],[8],[9]). Recently Aziz and Zargar [2] relaxed the hypothesis in several ways and proved

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$,

$$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 .$$

Then all the zeros of P(z) lie in

$$|z+k-1| \le \frac{ka_n + |a_0| - a_0}{|a_n|} .$$

For polynomials ,whose coefficients are not necessarily real, Govil and Rehman [6] proved the following generalisation of Theorem A:

Theorem C: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n, such that

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0 \ge 0 ,$$

where $\alpha_n > 0$, then P(z) has all its zeros in

$$|z| \le 1 + (\frac{2}{\alpha_n})(\sum_{j=0}^{n} |\beta_j|) .$$

More recently, Govil and Mc-tume [5] proved the following generalisations of Theorems B and C:

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some $k \ge 1$,

$$k\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0 ,$$

then P(z) has all its zeros in

$$|z+k-1| \le \frac{k\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|} .$$

Theorem E: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some $k\geq 1$,

$$k\beta_n \geq \beta_{n-1} \geq ......\geq \beta_1 \geq \beta_0,$$

then P(z) has all its zeros in

$$|z+k-1|\leq \frac{k\beta_n-\beta_0+|\beta_0|+2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|}.$$

M.H.Gulzar [7] proved the following generalisations of Theorems D and E.

Theorem F: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some real number $\rho\geq 0$,

$$\rho+\alpha_n \geq \alpha_{n-1} \geq ......\geq \alpha_1 \geq \alpha_0,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{\alpha_n}\right|\leq \frac{\rho+\alpha_n+|\alpha_0|-\alpha_0+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

Theorem G: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some real number $\rho\geq 0$,

$$\rho+\beta_n \geq \beta_{n-1} \geq ......\geq \beta_1 \geq \beta_0,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{\beta_n}\right|\leq \frac{\rho+\beta_n+|\beta_0|-\beta_0+2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|}.$$

The aim of this paper is to give generalizations of Theorem F and G under less restrictive conditions on the coefficients. More precisely we prove the following :

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some real numbers $\rho\geq 0$, $0<\sigma\leq 1$,

$$\rho+\alpha_n \geq \alpha_{n-1} \geq ......\geq \alpha_1 \geq \sigma\alpha_0,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{\alpha_n}\right|\leq \frac{\rho+\alpha_n-\sigma(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

Remark 1: Taking $\sigma=1$ in Theorem 1, we get Theorem F. Taking $\rho=(k-1)\alpha_n$, $\sigma=1$, Theorem 1 reduces to Theorem D and taking $\rho=0$, $\alpha_0>0$ and $\sigma=1$, we get Theorem C.

Applying Theorem 1 to P(tz), we obtain the following result:

Corollary 1 : Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some real numbers $\rho\geq 0$, $0<\sigma\leq 1$ and t>0,

$$\rho+t^n\alpha_n \geq t^{n-1}\alpha_{n-1} \geq ......\geq t\alpha_1 \geq \sigma\alpha_0,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{t^{n-1}\alpha_n}\right| \le \frac{\rho + t^n\alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|t^j}{t^{n-1}|\alpha_n|}.$$

In Theorem 1 , if we take $\alpha_0 \ge 0$, we get the following result:

Corollary 2 : Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some real numbers $\rho \ge 0$,
$0<\sigma \le 1$,

$$\rho + \alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \sigma\alpha_0 \ge 0,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{\alpha_n}\right| \le 1 + \frac{\rho + 2(1-\sigma)\alpha_0 + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

If we take $\rho = \alpha_{n-1} - \alpha_n \ge 0$ in Theorem 1, we get the following result:

Corollary 3 : Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n, such that

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \sigma\alpha_0 > 0.$$

Then P(z) has all its zeros in

$$\left|z+\frac{\alpha_{n-1}}{\alpha_n}-1\right| \le \frac{\alpha_{n-1} + 2(1-\sigma)\alpha_0 + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

Taking $\sigma = 1$ in Cor.3, we get the following result:

Corollary 4 : Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n, such that

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0 > 0.$$

Then P(z) has all its zeros in

$$\left|z+\frac{\alpha_{n-1}}{\alpha_n}-1\right| \le \frac{\alpha_{n-1} + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

If we apply Theorem 1 to the polynomial –iP(z) , we easily get the following result:

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n., If for some real numbers $\rho \ge 0$, $0<\sigma \le 1$,

$$\rho + \beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \sigma\beta_0,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{\beta_n}\right| \le \frac{\rho + \beta_n - \sigma(|\beta_0| + \beta_0) + 2|\beta_0| + 2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|}.$$

On applying Theorem 2 to the polynomial P(tz), one gets the following result:

Corollary 5 : Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1,2,……,n.If for some real numbers $\rho \ge 0$, $0<\sigma \le 1$ and t>0,

$$\rho + t^n\beta_n \ge t^{n-1}\beta_{n-1} \ge ...... \ge t\beta_1 \ge \beta_0 ,$$

then P(z) has all its zeros in the disk

$$\left| z + \frac{\rho}{t^{n-1}\beta_n} \right| \le \frac{\rho + t^n\beta_n - \sigma(|\beta_0| + \beta_0) + 2|\beta_0| + 2\sum_{j=0}^{n}|\alpha_j| t^j}{t^{n-1}|\beta_n|} .$$

II. Proofs of the Theorems

Proof of Theorem 1.

Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_1 - \alpha_0)z + \alpha_0 - i\beta_n z^{n+1} + i(\beta_n - \beta_{n-1})z^n \\ &\quad + ... + i(\beta_1 - \beta_0)z + i\beta_0 \\ &= -\alpha_n z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ... + (\alpha_1 - \sigma\alpha_0)z \\ &\quad + (\sigma\alpha_0 - \alpha_0)z + \alpha_0 + i\{-\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n + ... + (\beta_1 - \beta_0)z + \beta_0\} . \end{aligned}$$

Then

$$|F(z)| = \left| \begin{matrix} -\alpha_n z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ... + (\alpha_1 - \sigma\alpha_0)z \\ + (\sigma\alpha_0 - \alpha_0)z + \alpha_0 + i\{-\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n + ... + (\beta_1 - \beta_0)z + \beta_0\} \end{matrix} \right|$$

$$\ge |z|^n \left\{ \begin{matrix} |\alpha_n z + \rho| - |\rho + \alpha_n - \alpha_{n-1}| - |\alpha_0|\frac{1}{|z|^n} - |\alpha_1 - \sigma\alpha_0|\frac{1}{|z|^{n-1}} \\ -(1-\sigma)|\alpha_0| - \sum_{j=2}^{n-1}|\alpha_j - \alpha_{j-1}|\frac{1}{|z|^{n-j}} \end{matrix} \right\}$$

$$-\left| -\beta_n z^{n+1} + ... + (\beta_1 - \beta_0)z + \beta_0 \right| .$$

Thus , for $|z| > 1$,

$$|F(z)| > |z|^n \left\{ \begin{matrix} |\alpha_n z + \rho| - (\rho + \alpha_n - \alpha_{n-1}) - |\alpha_0| - (\alpha_{n-1} - \alpha_{n-2}) - ...... - (\alpha_2 - \alpha_1) \\ -(\alpha_1 - \sigma\alpha_0) - (1-\sigma)|\alpha_0| \end{matrix} \right\}$$

$$-(|\beta_n| + |\beta_0|) - \sum_{j=1}^{n}(|\beta_j| + |\beta_{j-1}|)$$

$$= |z|^n \left[|\alpha_n z + \rho| - \left\{ \rho + \alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j| \right\} \right]$$

$$> 0$$

if

$$|\alpha_n z + \rho| > \rho + \alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=1}^{n}|\beta_j| .$$

Hence all the zeros of F(z) whose modulus is greater than 1 lie in the disk

$$\left| z + \frac{\rho}{\alpha_n} \right| \le \frac{\rho + \alpha_n - \sigma(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|} .$$

But those zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence it follows that all the zeros of F(z) lie in the disk

$$\left|z+\frac{\rho}{\alpha_n}\right| \leq \frac{\rho+\alpha_n-\sigma(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

Since all the zeros of P(z) are also the zeros of F(z) ,it follows that all the zeros of P(z) lie in the disk

$$\left|z+\frac{\rho}{\alpha_n}\right| \leq \frac{\rho+\alpha_n-\sigma(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

This completes the proof of Theorem 1.

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