EXTREMALLY β-DISCONNECTEDNESS IN SMOOTH FUZZY β-CENTERED SYSTEM

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Abstract: *In this paper, we introduce maximal smooth fuzzy β-centered system, the smooth fuzzy space θ(R). Also extremally β-disconnectedness in smooth fuzzy β-centered system and its properties are studied.*

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I. Introduction and Preliminaries

The concept of fuzzy set was introduced by Zadeh [8]. Since then the concept has invaded nearly all branches of mathematics. In 1985, a fuzzy topology on a set X was defined as a fuzzy subset T of the family I^X of fuzzy subsets of X satisfying three axioms, the basic properties of such a topology were represented by Sostak [6]. In 1992, Ramadan [4], studied the concepts of smooth topological spaces. The method of centered systems in the theory of topology was introduced in [3]. In 2007, the above concept was extended to fuzzy topological spaces by Uma, Roja and Balasubramanian [8]. In this paper, the method of β-centered system is studied in the theory of smooth fuzzy topology. The concept of extremally β-disconnectedness in maximal structure θ(R) of maximal smooth fuzzy β-centered system is introduced and its properties are studied.

Definition 1.1. [6]
A function T: $I^X \to I$ is called a smooth fuzzy topology on X if it satisfies the following conditions:

a) $T(\bar{0}) = T(\bar{1}) = 1$

b) $T(\mu_1 \wedge \mu_2) \geq T(\mu_1) \wedge T(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$

c) $T\left(\bigvee_{i\in\Gamma} \mu_i\right) \geq \bigwedge_{i\in\Gamma} T(\mu_i)$ For any $\{\mu_i\}_{i\in\Gamma} \in I^X$

The pair (X, T) is called a smooth fuzzy topological space.

Definition 1.2. [7]
Let R be a fuzzy Hausdroff space. A system p = { λ_α } of fuzzy open sets of R is called fuzzy centered system if any finite collection of fuzzy sets of the system has a non-zero intersection. The system p is called maximal fuzzy centered system or a fuzzy end if it cannot be included in any larger fuzzy centered system.

Definition 1.3. [7]
Let θ(R) denotes the collection of all fuzzy ends belonging to R. We introduce a fuzzy topology in θ(R) in the following way: Let P_λ be the set of all fuzzy ends that include λ as an element, where λ is a fuzzy open set of R. Now P_λ is a fuzzy neighbourhood of each fuzzy end contained in P_λ. Thus to each fuzzy open set of R, there corresponds a fuzzy neighbourhood P_λ in θ(R).

Definition 1.4. [7]
A fuzzy Hausdroff space R is extremally disconnected if the closure of an open set is open.

Definition 1.5. [1]
The fuzzy real line R(L) is the set of all monotone decreasing elements $\lambda \in L^R$ satisfying $\vee \{ \lambda(t) / t \in R \} = 1$ and $\wedge \{ \lambda(t) / t \in R \} = 0$, after the identification of $\lambda, \mu \in L^R$ iff $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(t+)$ for all $t \in R$, where $\lambda(t-) = \wedge \{ \lambda(s) : s < t \}$ and $\lambda(t+) = \vee \{ \lambda(s) : s > t \}$. The natural L-fuzzy topology on R(L) is generated from the sub-basis $\{ L_t, R_t \}$ where $L_t(\lambda) = \lambda(t-)'$ and $R_t(\lambda) = \lambda(t+)$.

Definition 1.6. [2]
The L-fuzzy unit interval I (L) is a subset of R(L) such that $[\lambda] \in I(L)$ if $\lambda(t) = 1$ for $t < 0$ and $\lambda(t) = 0$ for $t > 1$.

Definition 1.7. [5]

A fuzzy set λ is quasi-coincident with a fuzzy set μ, denoted by $\lambda\, q\, \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$, otherwise $\lambda\, \not{q}\, \mu$.

II. The Spaces of maximal smooth fuzzy β-centered systems

In this section the maximal smooth fuzzy centered system is introduced and its properties are discussed.

Definition 2.1.

A smooth fuzzy topological space (X, T) is said to be smooth fuzzy β-Hausdorff iff for any two distinct fuzzy points x_{t_1}, x_{t_2} in X, there exists r-fuzzy β-open sets $\lambda, \mu \in I^X$ such that $x_{t_1} \in \lambda$ and $x_{t_2} \in \mu$ with $\lambda\, q\, \mu$.

Definition 2.2.

Let R be a smooth fuzzy β-Hausdorff space. A system $p_\beta = \{ \lambda_i \}$ of r-fuzzy β-open sets of R is called a smooth fuzzy β-centered system if any finite collection of $\{ \lambda_i \}$ is such that $\lambda_i \not{q} \lambda_j$ for $i \neq j$. The system p_β is called maximal smooth fuzzy β-centered system or a smooth fuzzy β-end if it cannot be included in any larger smooth fuzzy β-centered system.

Definition 2.3.

Let (X, T) be a smooth fuzzy topological space. Its Q*β-neighbourhood structure is a mapping $Q^* : X \times I^X \to I$ (X denotes the totality of all fuzzy points in X), defined by

$Q^*(x_0^t , \lambda) = \sup \{ \mu : \mu$ is an r-fuzzy β-open set, $\mu \leq \lambda,\ x_0^t \in \mu \}$ and

$\lambda = \inf_{x_0^t q \lambda} Q^*(x_0^t , \lambda)$ is r-fuzzy β-open set.

We note the following Properties of maximal smooth fuzzy β-centred system.

(1) If $\lambda_i \in p_\beta$ (i = 1, 2, 3…n), then $\bigwedge_{i=1}^{n} \lambda_i \in p$.

Proof:

If $\lambda_i \in p_\beta$ (i = 1, 2, 3…n), then $\lambda_i \not{q} \lambda_j$ for $i \neq j$. If $\bigwedge_{i=1}^{n} \lambda_i \notin p_\beta$, then $p_\beta \cup \{ \bigwedge_{i=1}^{n} \lambda_i \}$ will be a larger smooth fuzzy β-end than p. This contradicts the maximality of p_β. Therefore, $\bigwedge_{i=1}^{n} \lambda_i \in p_\beta$.

(2) If $\overline{0} \neq \lambda < \mu$, $\lambda \in p_\beta$ and μ is an r-fuzzy β-open set, then $\mu \in p_\beta$.

Proof:

If $\mu \notin p_\beta$, then $p_\beta \cup \{ \mu \}$ will be a larger smooth fuzzy β-end than p_β. This contradicts the maximality of p_β. Therefore $\mu \in p_\beta$.

(3) If λ is r-fuzzy β-open set, then $\lambda \notin p_\beta$ iff there exists $\mu \in p_\beta$ such that $\lambda\, q\, \mu$.

Proof:

Let $\lambda \notin p_\beta$ be an r-fuzzy β-open set. If there exists no $\mu \in p_\beta$ such that $\lambda\, q\, \mu$, then $\lambda\, \not{q}\, \mu$ for all $\mu \in p_\beta$. That is, $p_\beta \cup \{ \lambda \}$ will be a larger smooth fuzzy β-end than p_β. This contradicts the maximality of p_β.

Conversely, suppose that there exists $\mu \in p_\beta$ such that $\lambda\, q\, \mu$. If $\lambda \in p_\beta$, then $\lambda \not{q} \mu$. Contradiction. Hence $\lambda \notin p_\beta$.

(3) If $\lambda_1 \vee \lambda_2 = \lambda_3 \in p_\beta$, λ_1 and λ_2 are r-fuzzy β-open sets in R with $\lambda_1\, q\, \lambda_2$, then either $\lambda_1 \in p_\beta$ or $\lambda_2 \in p_\beta$.

Proof:

Let us suppose that both $\lambda_1 \in p_\beta$ and $\lambda_2 \in p_\beta$. Then $\lambda_1 \not{q} \lambda_2$. Contradiction. Hence either $\lambda_1 \in p_\beta$ or $\lambda_2 \in p_\beta$.

Note 2.1

Every smooth fuzzy β-centered system can be extended in atleast one way to a maximum one.

III. The Smooth Fuzzy maximal structure in θ(R).

In this section, smooth fuzzy maximal structure in the collection of all smooth fuzzy β-ends θ(R) is introduced and its properties are investigated.

Let θ(R) denotes the collection of all smooth fuzzy β-ends belonging to R. We introduce a smooth fuzzy maximal structure in θ(R) in the following way:

Let P_λ be the set of all smooth fuzzy β-ends that include λ as an element, where λ is a r-fuzzy β-open set of R. Now, P_λ is a smooth fuzzy Q* β-neighbourhood structure of each smooth fuzzy β-end contained in P_λ. Thus to each r-fuzzy β-open set λ of R corresponds a smooth fuzzy Q* β-neighbourhood structure P_λ in θ(R).

Proposition 3.1.
If λ and μ are r-fuzzy β-open sets, then
(a) $P_{\lambda\vee\mu} = P_\lambda \cup P_\mu$.
(b) $P_\lambda \cup P_{\bar{1}-C_{T(R)}(\lambda,r)} = \theta(R)$.

Proof:
(a) Let $p_\beta \in P_\lambda$. That is, $\lambda \in p_\beta$. Then by Property (2), $\lambda \vee \mu \in p_\beta$. That is, $p_\beta \in P_{\lambda\vee\mu}$. Hence $P_\lambda \cup P_\mu \subseteq P_{\lambda\vee\mu}$. Let $p\beta \in P_{\lambda\vee\mu}$ .That is, $\lambda \vee \mu \in p_\beta$. By the definition of P_λ, $\lambda \in p_\beta$ or $\mu \in p_\beta$. That is, $p_\beta \in P_\lambda$ or $p_\beta \in P_\mu$, therefore, $p_\beta \in P_\lambda \cup P_\mu$. This shows that $P_\lambda \cup P_\mu \supseteq P_{\lambda\vee\mu}$. Hence, $P_{\lambda\vee\mu} = P_\lambda \cup P_\mu$.
(b) If $p_\beta \notin P_{\bar{1}-C_{T(R)}(\lambda,r)}$, then $\bar{1} - C_{T(R)}(\lambda, r) \notin p_\beta$. That is, $\lambda \in p_\beta$ and $p_\beta \in P_\lambda$. Hence, $\theta(R) - P_{\bar{1}-C_{T(R)}(\lambda,r)} \subset P_\lambda$. If $p_\beta \in P_\lambda$, then $\lambda \in p_\beta$. That is, $\bar{1} - C_{T(R)}(\lambda, r) \notin p_\beta$, $p_\beta \notin P_{\bar{1}-C_{T(R)}(\lambda,r)}$. Therefore, $p_\beta \in \theta(R) - P_{\bar{1}-C_{T(R)}(\lambda,r)}$. That is, $P_\lambda \subset \theta(R) - P_{\bar{1}-C_{T(R)}(\lambda,r)}$. Hence, $P_\lambda \cup P_{\bar{1}-C_{T(R)}(\lambda,r)} = \theta(R)$.

Proposition 3.2.
θ(R) With the smooth fuzzy maximal structure described above is a smooth fuzzy β-compact space and has a base of smooth fuzzy Q*β-neighbourhood structure $\{P_\lambda\}$ that are both r-fuzzy β-open and r-fuzzy β-closed.

Proof:
Each P_λ in θ(R) is a r-fuzzy β-open by definition and by (b) of Proposition 3.1, it follows that it is r-fuzzy β-closed. Thus θ(R) has a base of smooth fuzzy Q*β-neighbourhood structure $\{P_\lambda\}$ that are both r-fuzzy β-open and r-fuzzy β-closed. We now show that θ(R) is smooth fuzzy β-compact. Let $\{P_{\lambda_\alpha}\}$ be a covering of θ(R) where each P_{λ_α} is r-fuzzy β-open. If it is impossible to pick a finite sub covering from the covering, then no set of the form $\bar{1} - \bigvee_{i=1}^{n} \beta\text{-}C_{T(R)}(\lambda_{\alpha_i}, r)$ is $\bar{0}$, since otherwise the sets $P_{\lambda_{\alpha_i}}$ would form a finite covering of θ(R). Hence the sets $\bar{1} - \bigvee_{i=1}^{n} \beta\text{-}C_{T(R)}(\lambda_{\alpha_i}, r)$ form a smooth fuzzy β-centered system. It may be extended to a maximal smooth fuzzy β-centered system p_β. This maximal smooth fuzzy β-centered system is not contained in $\{P_{\lambda_\alpha}\}$ since it contains in particular, all the $\bar{1} - \beta\text{-}C_{T(R)}(\lambda_{\alpha_i}, r)$. This contradiction proves that θ(R) is smooth fuzzy β-compact.

IV. Smooth fuzzy Extremally β-Disconnectedness in the maximal structure θ(R).

Definition 4.1.
A smooth fuzzy β-Hausdorff space R is smooth fuzzy extremally β-disconnected if $\beta\text{-}C_{T(R)}(\lambda, r)$ is r-fuzzy β-open for any r-fuzzy β-open set λ, $r \in I_0$.

Proposition 4.1.
The maximal smooth fuzzy structure θ(R) of maximal smooth fuzzy β-centered system of R is smooth fuzzy extremally β-disconnected.

Proof:
The proof of this theorem follows from the following equation $P_{\underset{\alpha}{\vee}\lambda_\alpha} = \beta\text{-}C_{T(\theta(R))}(\underset{\alpha}{\cup} P_{\lambda_\alpha}, r)$, $r \in I_0$. If $\lambda < \mu$, it follows that $P_\lambda \subset P_\mu$ and therefore $\underset{\alpha}{\cup} P_{\lambda_\alpha} \subset \beta\text{-}C_{T(\theta(R))}(P_{\underset{\alpha}{\vee}\lambda_\alpha}, r)$. By Proposition 3.2, $P_{\underset{\alpha}{\vee}\lambda_\alpha}$ is r-fuzzy β-closed and therefore, $\beta\text{-}C_{T(\theta(R))}(\underset{\alpha}{\cup} P_{\lambda_\alpha}, r) \subset P_{\underset{\alpha}{\vee}\lambda_\alpha}$. Let p be an arbitrary element of $P_{\underset{\alpha}{\vee}\lambda_\alpha} = \underset{\alpha}{\cup} P_{\lambda_\alpha}$. Then by Pro.3.1 (a), $p_\beta \in \beta\text{-}C_{T(\theta(R))}(\underset{\alpha}{\cup} P_{\lambda_\alpha}, r)$. Therefore, $P_{\underset{\alpha}{\vee}\lambda_\alpha} \subset \beta\text{-}C_{T(\theta(R))}(\underset{\alpha}{\cup} P_{\lambda_\alpha}, r)$. Hence, $P_{\underset{\alpha}{\vee}\lambda_\alpha} = \beta\text{-}C_{T(\theta(R))}(\underset{\alpha}{\cup} P_{\lambda_\alpha}, r)$.

Note 4.1.

The maximal structure θ(R) of maximal smooth fuzzy β-centered system is smooth fuzzy extremally β-disconnected if $P_{\vee_\alpha \lambda_\alpha}$ = β-$C_{T(\theta(R))}$($\bigcup_\alpha P_{\lambda_\alpha}$, r) where λ_α's r-fuzzy β-open sets. By Pro 3.1(a), it follows that $P_{\vee_\alpha \lambda_\alpha}$ = β-$C_{T(\theta(R))}$($P_{\vee_\alpha \lambda_\alpha}$, r). That is, P_{λ_Δ} = β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) where $\lambda_\Delta = \vee_\alpha \lambda_\alpha$.

Proposition 4.2.

Let θ(R) be an maximal smooth fuzzy β-centered system of the smooth fuzzy β-Hausdorff space R. Then the following conditions are equivalent:

(a) The space θ(R) is smooth fuzzy extremally β-disconnected.

(b) For each r-fuzzy β-open P_{λ_Δ}, β-$I_{T(\theta(R))}$(θ(R) – P_{λ_Δ},r) is r-fuzzy β-closed, r ∈ I_0.

(c) For each r-fuzzy β-open P_{λ_Δ}, β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + β-$C_{T(\theta(R))}$(θ(R) – $C_{T(\theta(R))}$(P_{λ_Δ}, r), r) = θ(R), r ∈ I_0.

(d) For every pair of collections of r-fuzzy β-open sets { P_{λ_Δ} } and { P_{μ_Δ} } such that β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + P_{μ_Δ} = θ(R), we have β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + β-$C_{T(\theta(R))}$(P_{μ_Δ}, r) = θ(R), r ∈ I_0.

Proof:

(a) ⇒ (b).

Let θ(R) be an smooth fuzzy extremally β-disconnected space and suppose that P_{λ_Δ} be r-fuzzy β-open, r ∈ I_0. Now, β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) = θ(R) – β-$I_{T(\theta(R))}$(θ(R) – P_{λ_Δ}, r). Since θ(R) is smooth fuzzy extremally β-disconnected, P_{λ_Δ} = β-$C_{T(\theta(R))}$(P_{λ_Δ}, r). Now, P_{λ_Δ} = θ(R) – β-$I_{T(\theta(R))}$(θ(R) – P_{λ_Δ}, r). Since , P_{λ_Δ} is r-fuzzy β-open.

(b) ⇒ (c).

Suppose that P_{λ_Δ} be r-fuzzy β-open, r ∈ I_0. Then,

β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + β-$C_{T(\theta(R))}$(θ(R) – β-$C_{T(\theta(R))}$(P_{λ_Δ}, r), r)
= β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + β -$C_{T(\theta(R))}$(β -$I_{T(\theta(R))}$(θ(R) – P_{λ_Δ}, r), r)
= β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + β-$I_{T(\theta(R))}$(θ(R) – P_{λ_Δ}, r)
= β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + θ(R) – β-$C_{T(\theta(R))}$(P_{λ_Δ}, r)
= θ(R).

(c) ⇒ (d).

Suppose that P_{λ_Δ} and P_{μ_Δ} are r-fuzzy β-open, r ∈ I_0, with

β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + P_{μ_Δ} = θ(R) (4.3.1)

Now by (c), we have

θ(R) = β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + β-$C_{T(\theta(R))}$(θ(R) – β-$C_{T(\theta(R))}$(P_{λ_Δ}, r), r)
= β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + β-$C_{T(\theta(R))}$(P_{μ_Δ}, r) (from (4.3.1))

Hence, β -$C_{T(\theta(R))}$(P_{λ_Δ}, r) + β -$C_{T(\theta(R))}$(P_{μ_Δ}, r) = θ(R).

(d) ⇒ (a).

Let us suppose that P_{μ_Δ} r-fuzzy β-open, r ∈ I_0 and let

P_{λ_Δ} = θ(R) – β-$C_{T(\theta(R))}$(P_{μ_Δ}, r) (4.3.2)

This implies that P_{λ_Δ} is r-fuzzy β-open. By (d), we have

β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) + β-$C_{T(\theta(R))}$(P_{μ_Δ}, r) = θ(R). (4.3.3)

From (4.3.2) and (4.3.3) we have,

P_{λ_Δ} = β-$C_{T(\theta(R))}$(P_{λ_Δ}, r). By Note 4.1, it follows that θ(R) is smooth fuzzy extremally β-disconnected.

Proposition 4.3.

Let θ(R) be the space of maximal smooth fuzzy β-centered system of the smooth fuzzy β-Hausdorff space R. Then, θ(R) is smooth fuzzy extremally β-disconnected iff for all r-fuzzy β-open P_{λ_Δ} and r-fuzzy β-closed P_{μ_Δ} with $P_{\lambda_\Delta} \subseteq P_{\mu_\Delta}$, β-$C_{T(\theta(R))}$($P_{\lambda_\Delta}$, r) ⊆ β-$I_{T(\theta(R))}$($P_{\mu_\Delta}$, r), r ∈ I_0.

Proof:

Let P_{λ_Δ} be r-fuzzy β-open and P_{μ_Δ} be r-fuzzy β-closed, r ∈ I_0, with $P_{\lambda_\Delta} \subseteq P_{\mu_\Delta}$. Then β-$I_{T(\theta(R))}$(P_{λ_Δ}, r) ⊆ β-$I_{T(\theta(R))}$(P_{μ_Δ}, r). That is, P_{λ_Δ} ⊆ β-$I_{T(\theta(R))}$(P_{μ_Δ}, r). This implies that, β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) ⊆ β-$C_{T(\theta(R))}$(β-$I_{T(\theta(R))}$(P_{μ_Δ}, r), r). By Proposition 4.2.(b), it follows that, β-$C_{T(\theta(R))}$(P_{λ_Δ}, r) ⊆ $I_{T(\theta(R))}$(P_{λ_Δ}, r).

Conversely, suppose that P_{μ_Δ} be r-fuzzy β-closed, $r \in I_0$. Then, $\beta\text{-}I_{T(\theta(R))}(P_{\mu_\Delta}, r) \subseteq P_{\mu_\Delta}$. By assumption, $\beta\text{-}C_{T(\theta(R))}(\beta\text{-}I_{T(\theta(R))}(P_{\mu_\Delta}, r), r) \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\mu_\Delta}, r)$ (4.3.1)

But, $\beta\text{-}I_{T(\theta(R))}(P_{\mu_\Delta}, r) \subseteq \beta\text{-}C_{T(\theta(R))}(\beta\text{-}I_{T(\theta(R))}(P_{\mu_\Delta}, r), r)$ (4.3.2)

From (4.3.1) and (4.3.2), we get

$\beta\text{-}I_{T(\theta(R))}(P_{\mu_\Delta}, r) = \beta\text{-}C_{T(\theta(R))}(\beta\text{-}I_{T(\theta(R))}(P_{\mu_\Delta}, r), r)$.

That is, $\beta\text{-}I_{T(\theta(R))}(P_{\mu_\Delta}, r))$ is r-fuzzy β-closed. By Proposition 4.2(b), it follows that θ(R) is smooth fuzzy extremally β-disconnected.

Remark 4.1.

Let θ(R) be an smooth fuzzy extremally β-disconnected space. Let $\{ P_{\lambda_{\Delta_i}}, \theta(R) - P_{\mu_{\Delta_i}}, I \varepsilon N \}$ be a collection such that $P_{\lambda_{\Delta_i}}$ are r-fuzzy β-open and $P_{\mu_{\Delta_I}}$ are r-fuzzy β-closed, $r \in I_0$. Let P_{λ_Δ}, P_{μ_Δ} are both r-fuzzy β-open and r-fuzzy β-closed. If $P_{\lambda_{\Delta_i}} \subseteq P_{\lambda_\Delta} \subseteq P_{\mu_{\Delta_i}}$ and $P_{\lambda_{\Delta_i}} \subseteq P_{\mu_\Delta} \subseteq P_{\mu_{\Delta_i}}$, then there exists an P_{η_Δ} which is both r-fuzzy β-open and r-fuzzy β-closed such that $\beta\text{-}C_{T(\theta(R))}(P_{\lambda_{\Delta_i}}, r) \subseteq P_{\eta_\Delta} \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\mu_{\Delta_i}}, r)$.

Proof:

By proposition 4.3, we have $\beta\text{-}C_{T(\theta(R))}(P_{\lambda_{\Delta_i}}, r) \subseteq \beta\text{-}C_{T(\theta(R))}(P_{\lambda_\Delta}, r) \cap \beta\text{-}I_{T(\theta(R))}(P_{\mu_\Delta}, r) \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\mu_{\Delta_i}}, r)$. Therefore, $P_{\eta_\Delta} = \beta\text{-}C_{T(\theta(R))}(P_{\lambda_\Delta}, r) \cap \beta\text{-}I_{T(\theta(R))}(P_{\mu_\Delta}, r)$ is such that r-fuzzy β-open and r-fuzzy β-closed. Hence $\beta\text{-}C_{T(\theta(R))}(P_{\lambda_{\Delta_i}}, r) \subseteq P_{\eta_\Delta} \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\mu_{\Delta_i}}, r)$.

Proposition 4.4.

Let θ(R) be an smooth fuzzy extremally β-disconnected space. Let $\{ P_{\lambda_{\Delta_q}} \}_{q\in Q}$ and $\{ P_{\mu_{\Delta_q}} \}_{q\in Q}$ be monotone increasing collections of r-fuzzy β-open and r-fuzzy β-closed sets and suppose that $P_{\lambda_{\Delta q_1}} \subseteq P_{\mu_{\Delta q_2}}$ whenever $q_1 < q_2$ (Q is the set of all rational numbers). Then there exists a monotone increasing collections $\{ P_{\eta_{\Delta q}} \}_{q\in Q}$ of r-fuzzy β-open and r-fuzzy β-closed sets such that $\beta\text{-}C_{T(\theta(R))}(P_{\lambda_{\Delta q_1}}, r) \subseteq P_{\eta_{\Delta q_2}}$ and $P_{\eta_{\Delta q_1}} \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\mu_{\Delta q_2}}, r)$ whenever $q_1 < q_2$, for all r-fuzzy β-open sets $\lambda_{\Delta q}, \mu_{\Delta q}, \eta_{\Delta q}, r \in I_0$.

Proof:

Let us arrange into a sequence $\{ q_n \}$ of all rational numbers (without repetition). For every $n \geq 2$, we shall define inductively a collection $\{ P_{\eta_{\Delta q_i}} / 1 \leq i \leq n \}$ such that for all $i < n$

$$\left.\begin{array}{ll} \beta\text{-}C_{T(\theta(R))}(P_{\lambda_{\Delta q}}, r) \subseteq P_{\eta_{\Delta q_i}} & \text{if } q < q_i \\ P_{\eta_{\Delta q_i}} \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\mu_{\Delta q}}, r) & \text{if } q_i < q \end{array}\right\} \quad (S_n)$$

By Proposition 4.3.3, the countable collection $\{ \beta\text{-}C_{T(\theta(R))}(P_{\lambda_{\Delta q_1}}, r)\}$ and $\{\beta\text{-}I_{T(\theta(R))}(P_{\mu_{\Delta q_2}}, r)\}$ satisfy $\beta\text{-}C_{T(\theta(R))}(P_{\lambda_{\Delta q_1}}, r) \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\mu_{\Delta q_2}}, r)$ if $q_1 < q_2$. By Remark 4.3.1., there exists $P_{\delta_{\Delta_1}}$ which is both r-fuzzy β-open and r-fuzzy β-closed, with $\beta\text{-}C_{T(\theta(R))}(P_{\lambda_{\Delta q_1}}, r) \subseteq P_{\delta_{\Delta_1}} \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\mu_{\Delta q_2}}, r)$. Setting $P_{\delta_{\Delta_1}} = P_{\eta_{\Delta q_1}}$ we get (S_2). Define $P_{\psi_\Delta} = \cup \{ P_{\eta_{\Delta q_i}} / i < n, q_i < q_n \} \cup P_{\lambda_{\Delta q_n}}$ and $P_{\phi_\Delta} = \cap \{ P_{\eta_{\Delta q_j}} / j < n, q_j > q_n \} \cap P_{\mu_{\Delta q_n}}$. Then, we have $\beta\text{-}C_{T(\theta(R))}(P_{\eta_{\Delta q_i}}, r) \subseteq \beta\text{-}C_{T(\theta(R))}(P_{\psi_\Delta}, r) \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\eta_{\Delta q_j}}, r)$ and $\beta\text{-}C_{T(\theta(R))}(P_{\eta_{\Delta q_i}}, r) \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\phi_\Delta}, r) \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\eta_{\Delta q_j}}, r)$ whenever $q_i < q_n < q_j$ $(i < j < n)$ and $P_{\lambda_{\Delta q}} \subseteq \beta\text{-}C_{T(\theta(R))}(P_{\psi_\Delta}, r) \subseteq P_{\mu_{\Delta q'}}$ and $P_{\lambda_{\Delta q}} \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\phi_\Delta}, r) \subseteq P_{\mu_{\Delta q'}}$ whenever $q < q_n < q'$. This shows that the countable collections $\{P_{\eta_{\Delta q_i}} / i < n, q_i < q_n \} \cup \{P_{\lambda_{\Delta q}} / q < q_n \}$ and $\{P_{\eta_{\Delta q_j}} / j < n, q_j > q_n \} \cup \{P_{\mu_{\Delta q}} / q > q_n \}$ together with P_{ψ_Δ} and P_{ϕ_Δ} fulfill all the conditions of Remark 4.1. Hence there exists a collection $P_{\delta_{\Delta q_n}}$ which is r-fuzzy β-open and r-fuzzy β-closed such that

$\beta\text{-}C_{T(\theta(R))}(P_{\delta_{\Delta q_n}}, r) \subseteq P_{\mu_{\Delta q}}$ if $q > q_n$,

$P_{\lambda_{\Delta q}} \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\delta_{\Delta q_n}}, r)$ if $q < q_n$

$\beta\text{-}C_{T(\theta(R))}(P_{\eta_{\Delta q_i}}, r) \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\delta_{\Delta q_n}}, r)$ if $q_i < q_n$

$\beta\text{-}C_{T(\theta(R))}(P_{\delta_{\Delta q_n}}, r) \subseteq \beta\text{-}I_{T(\theta(R))}(P_{\eta_{\Delta q_j}}, r)$ if $q_j > q_n$ where $1 \leq i, j \leq n - 1$.

Now setting $P_{\eta_{\Delta q_n}} = P_{\delta_{\Delta q_n}}$ we obtain the collections $P_{\eta_{\Delta q_1}}, P_{\eta_{\Delta q_2}}, \ldots, P_{\eta_{\Delta q_n}}$, that satisfy (S_{n+1}). Therefore the collection $\{ P_{\eta_{\Delta q_i}} / i = 1,2,3, \text{---} n \}$ has the required property.

Definition 4.2.

Let θ(R) be an maximal smooth fuzzy β-centered system. The smooth fuzzy real line R*(I) in smooth fuzzy β-centered system is the set of all monotone decreasing r-fuzzy β-sets $\{ P_{\lambda_\Delta} \}$ satisfying $\cup \{ P_{\lambda_\Delta(t)} / t \in R \} = \theta(R)$ and $\cap \{ P_{\lambda_\Delta(t)} / t \in R \} = \phi$, after the identification of P_{λ_Δ} and P_{μ_Δ} iff $P_{\lambda_\Delta(t-)} = P_{\mu_\Delta(t-)}$ and $P_{\lambda_\Delta(t+)} = P_{\mu_\Delta(t+)}$ for all $t \in R$, where $P_{\lambda_\Delta(t-)} = \cap \{ P_{\lambda_\Delta(s)} / s < t \}$ and $P_{\lambda_\Delta(t+)} = \cup \{ P_{\lambda_\Delta(s)} / s > t \}$. The natural smooth fuzzy topology on R*(I) is generated from the sub-basis $\{ L_t^*, R_t^* \}$ where $L_t^*[P_{\lambda_\Delta}] = P_{\lambda_\Delta(t-)}$ and $R_t^*[P_{\lambda_\Delta}] = P_{\lambda_\Delta(t+)}$. A partial order on R*(I) is defined by $[P_{\lambda_\Delta}] \le [P_{\mu_\Delta}]$ iff $P_{\lambda_\Delta(t-)} \subseteq P_{\mu_\Delta(t-)}$ and $P_{\lambda_\Delta(t+)} \subseteq P_{\mu_\Delta(t+)}$ for all $t \in R$.

Definition 4.3.

Let θ(R) be an maximal smooth fuzzy β-centered system. The smooth fuzzy unit interval I*(I) in smooth fuzzy β-centered system is a subset of R*(I) such that $[P_{\lambda_\Delta}] \in I^*(I)$ if $P_{\lambda_\Delta(t)} = \theta(R)$ for $t < 0$ and $P_{\lambda_\Delta(t)} = \phi$ for $t > 1$ where λ_Δ's are r-fuzzy β-open set and $t \in R$, $r \in I_0$.

Definition 4.4.

Let θ(R) be an maximal smooth fuzzy β-centered system. A mapping $f : \theta(R) \to R^*(I)$ is called lower (upper) smooth fuzzy β-continuous if $f^{-1}(R_t^*)$ (resp. $f^{-1}(L_t^*)$) is r-fuzzy β-open (resp. $f^{-1}(L_t^*)$ is r-fuzzy β-open and r-fuzzy β-closed set), for all $t \in R$, $r \in I_0$.

Proposition 4.5.

Let θ(R) be an maximal smooth fuzzy β-centered system. Let $f : \theta(R) \to R^*(I)$ be a mapping such that

$$f(P_{\lambda_\Delta(t)}) = \begin{cases} \theta(R) & t < 0 \\ P_{\lambda_\Delta(t)} & 0 \le t \le 1 \\ \phi & t > 1 \end{cases}$$

Where λ_Δ is a r-fuzzy β-open set. Then f is lower (upper) smooth fuzzy β-continuous iff λ_Δ is a r-fuzzy β-open set(resp r-fuzzy β-closed).

Proof:

Now,

$$f^{-1}(R_t^*) = \begin{cases} \theta(R) & t < 0 \\ & 0 \le t \le 1 \\ \phi & t > 1 \end{cases}$$

implies that f is lower smooth fuzzy β-continuous iff $P_{\lambda(t)}$ is r-fuzzy β-open.

Now,

$$f^{-1}(L_t^*) = \begin{cases} \theta(R) & t < 0 \\ P_{\lambda_\Delta(t)} & 0 \le t \le 1 \\ \phi & t > 1 \end{cases}$$

implies that f is upper smooth fuzzy β-continuous iff $P_{\lambda_\Delta(t)}$ is r-fuzzy β-open and r-fuzzy β-closed.

Definition 4.5.

Let θ(R) be an maximal smooth fuzzy β-centered system. The characteristic function $\chi_{P_{\lambda_\Delta}}(P_{\lambda_\Delta})$ is a function $\chi_{P_{\lambda_\Delta}} : \theta(R) \to I^*(I)$ defined by $\chi_{P_{\lambda_\Delta}}(P_{\mu_\Delta}) = P_{\lambda_\Delta}$ if $P_{\mu_\Delta} \in \theta(R)$.

Definition 4.6.

Let θ(R) be an maximal smooth fuzzy β-centered system. Then $\chi_{P_{\lambda_\Delta}}$ is lower (resp. upper) smooth fuzzy β-continuous iff P_{λ_Δ} is r-fuzzy β-open(resp., P_{λ_Δ} is r-fuzzy β-open and r-fuzzy β-closed), $r \in I_0$.

Definition 4.7.

Let $\theta(R)$ be an maximal smooth fuzzy β-centered system. Then $f : \theta(R) \to R^*(I)$ is said to be strongly smooth fuzzy β-continuous if $f^{-1}(R_t^*)$ is smooth fuzzy β-open and $f^{-1}(L_t^*)$ is both r-fuzzy β-open and r-fuzzy β-closed, for all $t \in R$, $r \in I_0$.

Proposition 4.7.

Let $\theta(R)$ be an maximal smooth fuzzy β-centered system. Then the following statements are equivalent :

(a) $\theta(R)$ is an smooth fuzzy extremally β-disconnected space.

(b) If $g, h : \theta(R) \to R^*(I)$, where g is lower smooth fuzzy β-continuous, h is upper smooth fuzzy β-continuous and $g \le h$, then there exists a strong smooth fuzzy β-continuous function f such that $g \le f \le h$.

(c) If $\theta(R) - P_{\lambda_\Delta}$ and P_{μ_Δ} are both r-fuzzy β-open and β-closed with $P_{\mu_\Delta} \subseteq P_{\lambda_\Delta}$, then there exist a strong smooth fuzzy β-continuous function $f : \theta(R) \to I$ such that $P_{\mu_\Delta} \subseteq (\theta(R) - L_1^*)f \subseteq R_0^*f \subseteq P_{\lambda_\Delta}$.

Proof:

(a) ⇒ (b)

Define $H_i = h^{-1}L_i^*$ and $G_i = g^{-1}(\theta(R) - R_i^*)$, $i \in Q$. Then we have two monotone increasing collections H_i which are r-fuzzy β-open sets and G_i r-fuzzy β-closed sets, $r \in I_0$. Moreover $H_i \subseteq G_j$ if $i < j$. By Proposition 4.3.4, there exists a monotone increasing collections of r-fuzzy β-open and r-fuzzy β-closed sets $\{ F_i \}_{i\in Q}$, such that $\beta\text{-}C_{T(\theta(R))}(H_i, r) \subseteq F_j$ and $F_i \subseteq \beta\text{-}I_{T(\theta(R))}(G_j, r)$ if $i < j$. Set $V_k = \bigcap_{i<k} (1 - F_i)$ such that V_k is a monotone decreasing collection of r-fuzzy β-open and r-fuzzy β-closed sets.

Moreover, $\beta\text{-}C_{T(\theta(R))}(V_k, r) \subseteq \beta\text{-}I_{T(\theta(R))}(V_j, r)$ whenever $k < j$.

Therefore,
$$\begin{aligned}\bigcup_{k\in R} V_k &= \bigcup_{k\in R} \Big(\bigcap_{i<k} (1 - F_i)\Big)\\ &\supseteq \bigcup_{k\in R} \Big(\bigcap_{i<k} (1 - G_i)\Big)\\ &= \bigcup_{k\in R} \Big(\bigcap_{i<k} g^{-1}(R_i^*)\Big)\\ &= \bigcup_{k\in R} \big(g^{-1}(R_k^*)\big)\\ &= g^{-1}\Big(\bigcup_{k\in R} R_k^*\Big)\\ &= \theta(R).\end{aligned}$$

Similarly, $\bigcap_{k\in R} V_k = \phi$.

Define a function $f : \theta(R) \to R^*(I)$ satisfying the required properties. Let $f(P_{\lambda_{\Delta_i}}) = \eta_{\Delta_i}(t)$ where $P_{\eta_{\Delta_i}(t)}$ is a collection in V_k. To prove that f is strongly smooth fuzzy β-continuous. We observe that $\bigcup_{j>k} V_j = \bigcup_{j>k} \beta\text{-}I_{T(\theta(R))}(V_j, r)$ and $\bigcap_{j<k} V_j = \bigcap_{j<k} \beta\text{-}C_{T(\theta(R))}(V_j, r)$. Then $f^{-1}(R_k^*) = \bigcup_{j>k} V_j = \bigcup_{j>k} \beta\text{-}I_{T(\theta(R))}(V_j, r)$ is r-fuzzy β-open set and $f^{-1}(1 - L_k^*) = \bigcap_{j<k} V_j = \bigcap_{j<k} \beta\text{-}C_{T(\theta(R))}(V_j, r)$ is r-fuzzy β-closed and $f^{-1}(L_k^*)$ is r-fuzzy β-open set. Hence f is strongly smooth fuzzy β-continuous.

To show that $g \le f \le h$. That is, $g^{-1}(1 - L_t^*) \subseteq f^{-1}(1 - L_t^*) \subseteq h^{-1}(1 - L_t^*)$, $g^{-1}(R_t^*) \subseteq f^{-1}(R_t^*) \subseteq h^{-1}(R_t^*)$.

Now,
$$\begin{aligned}g^{-1}(1 - L_t^*) &= \bigcap_{s<t} g^{-1}(1 - L_s^*)\\ &= \bigcap_{s<t} \bigcap_{p<s} g^{-1}(R_p^*)\\ &= \bigcap_{s<t} \bigcap_{p<s} (1 - G_p)\\ &\subseteq \bigcap_{s<t} \bigcap_{p<s} (1 - F_p)\\ &= \bigcap_{s<t} V_s\\ &= f^{-1}(1 - L_t^*)\end{aligned}$$

$$f^{-1}(\theta(R) - L_t^*) = \bigcap_{s<t} V_s$$

$$= \bigcap_{s<t} \bigcap_{p<s} (1 - F_p)$$

$$\subseteq \bigcap_{s<t} \bigcap_{p<s} (1 - H_p)$$

$$= \bigcap_{s<t} \bigcap_{p<s} h^{-1}(1 - L_p^*)$$

$$= \bigcap_{s<t} h^{-1}(1 - L_s^*)$$

$$= h^{-1}(1 - L_t^*)$$

Similarly we obtain,

$$g^{-1}(R_t^*) = \bigcup_{s>t} g^{-1}(R_s^*)$$

$$= \bigcup_{s>t} \bigcup_{p>s} g^{-1}(R_p^*)$$

$$= \bigcup_{s>t} \bigcup_{p>s} (1 - G_p)$$

$$\subseteq \bigcup_{s>t} \bigcap_{p<s} (1 - F_p)$$

$$= \bigcup_{s>t} V_s$$

$$= f^{-1}(R_t^*) \text{ and}$$

$$f^{-1}(R_t^*) = \bigcup_{s>t} V_s$$

$$= \bigcup_{s>t} \bigcap_{p<s} (1 - F_p)$$

$$\subseteq \bigcup_{s>t} \bigcap_{p<s} (1 - H_p)$$

$$= \bigcup_{s>t} \bigcap_{p<s} h^{-1}(1 - L_p^*)$$

$$= \bigcup_{s>t} h^{-1}(R_s^*)$$

$$= h^{-1}(R_t^*).$$

Thus (b) is proved.

(b) $\Rightarrow$ (c)

Suppose P_{λ_Δ} is r-fuzzy β-open set and r-fuzzy β-closed set and P_{μ_Δ} is r-fuzzy β-open set and r-fuzzy β-closed set with $P_{\mu_\Delta} \subseteq P_{\lambda_\Delta}$. Then $\chi_{P_{\mu_\Delta}} \subseteq \chi_{P_{\lambda_\Delta}}$, where $\chi_{P_{\mu_\Delta}}$, $\chi_{P_{\lambda_\Delta}}$ are lower and upper smooth fuzzy β-continuous function respectively. By (b), there exist a strongly smooth fuzzy β-continuous function $f : \theta(R) \to R(I)$ such that $\chi_{P_{\mu_\Delta}} \leq f \leq \chi_{P_{\lambda_\Delta}}$. Clearly $f(P_{\lambda_\Delta}) \in I^*(I)$ and $P_{\mu_\Delta} = (1 - L_1^*)\chi_{P_{\mu_\Delta}} \subseteq (1 - L_1^*)f \subseteq R_0^*f \subseteq R_0^* \chi_{P_{\lambda_\Delta}} \subseteq P_{\lambda_\Delta}$. Therefore, $P_{\mu_\Delta} \subseteq (1 - L_1^*)f \subseteq R_0^*f \subseteq P_{\lambda_\Delta}$.

(c) $\Rightarrow$ (a)

By (c), it follows that $(1 - L_1^*)f$ and R_0^*f are r-fuzzy β-open and r-fuzzy β-closed. By Proposition 4.3, it follows that θ(R) is an smooth fuzzy extremally β-disconnected space.

V. Tietze Extension Theorem

In this section, Tietze Extension Theorem for smooth fuzzy extremally β-disconnected space is discussed.

Proposition 5.1.

Let θ(R) be a smooth fuzzy extremally β-disconnected space. Let $A \subseteq \theta(R)$ and the collection $\{ P_{\lambda_\Delta} \}$ in A such that $\chi_{P_{\lambda_\Delta}}$ is r-fuzzy β-open. Let $f : A \to I^*(I)$ be a strongly smooth fuzzy β-continuous function. Then, f has a strongly smooth fuzzy β-continuous extension over θ(R).

Proof:
Let g, h : $\theta(R) \rightarrow I^*(I)$ be such that g = f = h on A.
Now,

$$R_t{}^*g = \begin{cases} P_{\mu_{\Delta_t}} \wedge \chi_{P_{\lambda_\Delta}} & \text{if } t \geq 0 \\ & \text{if } t < 0 \end{cases}$$

where $P_{\mu_{\Delta_t}}$ is r-fuzzy β-open set and is such that $P_{\mu_{\Delta_t}} = R_t{}^*g$ in A.

$$L_t{}^*h = \begin{cases} P_{\lambda_{\Delta_t}} \wedge \chi_{P_{\lambda_\Delta}} & \text{if } t \leq 1 \\ \theta(R) & \text{if } t > 1 \end{cases}$$

where $P_{\lambda_{\Delta_t}}$ is both r-fuzzy β-open and β-closed set is such that $P_{\lambda_{\Delta_t}} = L_t{}^*h$ in A. Thus g is lower smooth fuzzy β-continuous, h is upper smooth fuzzy β-continuous and $g \leq h$. By Proposition 4.7, there is a strong smooth fuzzy β-continuous function F: $\theta(R) \rightarrow I^*(I)$ such that $g \leq F \leq h$. Hence $f \equiv F$ on A.

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