Further Results On The Basis Of Cauchy's Proper Bound for the Zeros of Entire Functions of Order Zero

Sanjib Kumar Datta[1], Manab Biswas[2]
[1](Department of Mathematics, University of Kalyani, Kalyani, Dist-Nadia, Pin- 741235, West Bengal, India)
[2](Barabilla High School, P.O. Haptiagach, Dist-Uttar Dinajpur, Pin- 733202, West Bengal, India)

ABSTRACT: *A single valued function of one complex variable which is analytic in the finite complex plane is called an entire function . The purpose of this paper is to establish the bounds for the moduli of zeros of entire functions of order zero. Some examples are provided to clear the notions.*

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I. INTRODUCTION, DEFINITIONS AND NOTATIONS

Let, $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots\ldots + a_{n-1} z^{n-1} + a_n z^n$; $|a_n| \neq 0$
Be a polynomial of degree n . Datt and Govil [2] ; Govil and Rahaman [5] ; Marden [9] ; Mohammad [10] ; Chattopadhyay, Das, Jain and Konwar [1] ; Joyal, Labelle and Rahaman [6] ; Jain {[7] , [8]}; Sun and Hsieh [11] ; Zilovic, Roytman, Combettes and Swamy [13] ; Das and Datta [4] etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

In this paper we intend to establish some of sharper results concerning the theory of distribution of zeros of entire functions of order zero.

The following definitions are well known :

Definition 1 The order ρ and lower order λ of a meromorphic function f are defined as

$$\rho = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho = \limsup_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda = \liminf_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$.

If $\rho < \infty$ then f is of finite order. Also $\rho = 0$ means that f is of order zero. In this connection Datta and Biswas [3] gave the following definition :

Definition 2 Let f be a meromorphic function of order zero. Then the quantities ρ^* and λ^* of f are defined by :

$$\rho^* = \limsup_{r \to \infty} \frac{T(r,f)}{\log r} \quad \text{and} \quad \lambda^* = \liminf_{r \to \infty} \frac{T(r,f)}{\log r}.$$

If f is an entire function then clearly

$$\rho^* = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r} \quad \text{and} \quad \lambda^* = \liminf_{r \to \infty} \frac{\log M(r,f)}{\log r}.$$

II. LEMMAS

In this section we present a lemma which will be needed in the sequel .

Lemma 1 If $f(z)$ is an entire function of order $\rho = 0$, then for every $\varepsilon > 0$ the inequality $N(r) \leq (\log r)^{\rho^* + \varepsilon}$
Holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq \log r$.

Proof. Let us suppose that $f(z) = 1$. This supposition can be made without loss of generality because if $f(z)$ has a zero of order $'m'$ at the origin then we may consider $g(z) = c.\frac{f(z)}{z^m}$ where c is so chosen that $g(0) = 1$. Since the function $g(z)$ and $f(z)$ have the same order therefore it will be unimportant for our investigations that the number of zeros of $g(z)$ and $f(z)$ differ by m.

We further assume that $f(z)$ has no zeros on $|z| = \log 2r$ and the zeros z_i's of $f(z)$ in $|z| < \log r$ are in non decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also let ρ^* suppose to be finite where $\rho = 0$ is the zero of order of $f(z)$.

Now we shall make use of Jenson's formula as state below

$$\log|f(0)| = -\sum_{i=1}^{n} \log\frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(R\,e^{i\phi})|\,d\phi\ . \quad (1)$$

Let us replace R by $2r$ and n by $N(2r)$ in (1) .

$$\therefore \ \log|f(0)| = -\sum_{i=1}^{N(2r)} \log\frac{2r}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(2r\,e^{i\phi})|\,d\phi.$$

Since $f(0) = 1$, $\therefore \log|f(0)| = \log 1 = 0$.

$$\therefore \sum_{i=1}^{N(2r)} \log\frac{2r}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(2r\, e^{i\phi})\right| d\phi \tag{2}$$

$$\text{L.H.S.} = \sum_{i=1}^{N(2r)} \log\frac{2r}{|z_i|} \geq \sum_{i=1}^{N(r)} \log\frac{2r}{|z_i|} \geq N(r)\log 2 \tag{3}$$

because for large values of r,

$$\log\frac{2r}{|z_i|} \geq \log 2\ .$$

$$\text{R.H.S.} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(2r\, e^{i\phi})\right| d\phi$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \log M(2r)\, d\phi = \log M(2r). \tag{4}$$

Again by definition of order ρ^* of $f(z)$ we have for every $\varepsilon > 0$,

$$\log M(2r) \leq \{\log(2r)\}^{\rho^*+\varepsilon/2}\ . \tag{5}$$

Hence from (2) by the help of (3) , (4) and (5) we have

$$N(r)\log 2 \leq \{\log(2r)\}^{\rho^*+\varepsilon/2}$$

$$\text{i.e., } N(r) \leq \frac{(\log\ 2)^{\rho^*+\varepsilon/2}}{\log\ 2}\cdot\frac{(\log\ r)^{\rho^*+\varepsilon}}{(\log\ r)^{\varepsilon/2}} \leq (\log r)^{\rho^*+\varepsilon}\ .$$

This proves the lemma.

III. THEOREMS

In this section we present the main results of the paper.

Theorem 1 Let $P(z)$ be an entire function having order $\rho = 0$ in the disc $|z| \leq \log r$ for sufficiently large r. Also let the Taylor's series expansion of $P(z)$ be given by

$$P(z) = a_0 + a_{p_1}z^{p_1} + \ldots\ldots + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)},\ a_0 \neq 0,\ a_{N(r)} \neq 0$$

with $1 \leq p_1 < p_2 < \ldots\ldots < p_m \leq N(r) - 1$, p_i's are integers such that for some $\rho^* > 0$,

$$|a_0|(\rho^*)^{N(r)} \geq |a_{p_1}|(\rho^*)^{N(r)-p_1} \geq \ldots\ldots \geq |a_{p_m}|(\rho^*)^{N(r)-p_m} \geq |a_{N(r)}|\ .$$

Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^*\left(1+\frac{|a_{p_1}|}{|a_0|(\rho^*)^{p_1}}\right)} < |z| < \frac{1}{\rho^*}\left(1 + \frac{|a_0|}{|a_{N(r)}|}(\rho^*)^{N(r)}\right).$$

Proof. Given that

$$P(z) = a_0 + a_{p_1}z^{p_1} + \ldots\ldots + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)}$$

where p_i's are integers and $1 \leq p_1 < p_2 < \ldots\ldots < p_m \leq N(r) - 1$. Then for some $\rho^* > 0$,

$$|a_0|(\rho^*)^{N(r)} \geq |a_{p_1}|(\rho^*)^{N(r)-p_1} \geq \ldots\ldots \geq |a_{p_m}|(\rho^*)^{N(r)-p_m} \geq |a_{N(r)}|\ .$$

Let us consider

$$Q(z) = (\rho^*)^{N(r)}P\left(\frac{z}{\rho^*}\right)$$

$$= (\rho^*)^{N(r)}\left\{a_0 + a_{p_1}\frac{z^{p_1}}{(\rho^*)^{p_1}} + \cdots\cdots + a_{p_m}\frac{z^{p_m}}{(\rho^*)^{p_m}} + a_{N(r)}\frac{z^{N(r)}}{(\rho^*)^{N(r)}}\right\}$$

$$= a_0(\rho^*)^{N(r)} + a_{p_1}(\rho^*)^{N(r)-p_1}z^{p_1} + \ldots\ldots + a_{p_m}(\rho^*)^{N(r)-p_m}z^{p_m} + a_{N(r)}z^{N(r)}\ .$$

Therefore

$$|Q(z)| \geq |a_{N(r)}z^{N(r)}| - |a_0(\rho^*)^{N(r)} + a_{p_1}(\rho^*)^{N(r)-p_1}z^{p_1} + \ldots\ldots + a_{p_m}(\rho^*)^{N(r)-p_m}z^{p_m}|. \tag{6}$$

Now using the given condition of Theorem 1 we obtain that

$$|a_0(\rho^*)^{N(r)} + a_{p_1}(\rho^*)^{N(r)-p_1}z^{p_1} + \ldots\ldots + a_{p_m}(\rho^*)^{N(r)-p_m}z^{p_m}|$$

$$\leq |a_0|(\rho^*)^{N(r)} + |a_{p_1}|(\rho^*)^{N(r)-p_1}|z|^{p_1} + \ldots\ldots + |a_{p_m}|(\rho^*)^{N(r)-p_m}|z|^{p_m}$$

$$\leq |a_0|(\rho^*)^{N(r)}|z|^{N(r)}\left(\frac{1}{|z|^{N(r)-p_m}} + \cdots\cdots + \frac{1}{|z|^{N(r)}}\right) \text{ for } |z| \neq 0\ .$$

Using (6) we get for $|z| \neq 0$ that

$$|Q(z)| \geq |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}|z|^{N(r)}\left(\frac{1}{|z|^{N(r)-p_m}} + \cdots\cdots + \frac{1}{|z|^{N(r)}}\right)$$

$$> |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}|z|^{N(r)}\left(\frac{1}{|z|^{N(r)-p_m}} + \cdots\cdots + \frac{1}{|z|^{N(r)}} + \cdots\cdot\right)$$

$$= |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}|z|^{N(r)}\left(\sum_{k=1}^{\infty}\frac{1}{|z|^k}\right). \tag{7}$$

The geometric series $\sum_{k=1}^{\infty}\frac{1}{|z|^k}$ is convergent for

$$\frac{1}{|z|} < 1$$

$$\text{i.e., for } |z| > 1$$

and converges to

$$\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}.$$

Therefore

$$\sum_{k=1}^{\infty}\frac{1}{|z|^k} = \frac{1}{|z|-1} \text{ for } |z| > 1.$$

Using (7) we get from above that for $|z| > 1$

$$|Q(z)| \geq |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}|z|^{N(r)}\left(\frac{1}{|z|-1}\right)$$
$$= |z|^{N(r)}\left(|a_{N(r)}| - \frac{|a_0|(\rho^*)^{N(r)}}{|z|-1}\right).$$

Now for $|z| > 1$,

$$|Q(z)| \geq 0 \text{ if } |a_{N(r)}| - \frac{|a_0|(\rho^*)^{N(r)}}{|z|-1} \geq 0$$
$$\text{i.e., if } |a_{N(r)}| \geq \frac{|a_0|(\rho^*)^{N(r)}}{|z|-1}$$
$$\text{i.e., if } |z|-1 \geq \frac{|a_0|(\rho^*)^{N(r)}}{|a_{N(r)}|}$$
$$\text{i.e., if } |z| \geq 1 + \frac{|a_0|(\rho^*)^{N(r)}}{|a_{N(r)}|} > 1.$$

Therefore $|Q(z)| \geq 0$ if

$$|z| \geq 1 + \frac{|a_0|(\rho^*)^{N(r)}}{|a_{N(r)}|}.$$

Therefore $Q(z)$ does not vanish for

$$|z| \geq 1 + \frac{|a_0|(\rho^*)^{N(r)}}{|a_{N(r)}|}.$$

So all the zeros of $Q(z)$ lie in

$$|z| < 1 + \frac{|a_0|(\rho^*)^{N(r)}}{|a_{N(r)}|}.$$

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.
Putting $z = \rho^* z_0$ in $Q(z)$ we get that

$$Q(\rho^* z_0) = (\rho^*)^{N(r)}.P(z_0) = (\rho^*)^{N(r)}.0 = 0.$$

So $z = \rho^* z_0$ is a zero of $Q(z)$. Hence

$$|\rho^* z_0| < 1 + \frac{|a_0|(\rho^*)^{N(r)}}{|a_{N(r)}|}$$

$$\text{i.e., } |z_0| < \frac{1}{\rho^*}\left(1 + \frac{|a_0|(\rho^*)^{N(r)}}{|a_{N(r)}|}\right).$$

Since z_0 is an arbitrary zero of $P(z)$, therefore all the zeros of $Q(z)$ lie in

$$|z| < \frac{1}{\rho^*}\left(1 + \frac{|a_0|(\rho^*)^{N(r)}}{|a_{N(r)}|}\right). \tag{8}$$

Again let us consider

$$R(z) = (\rho^*)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^* z}\right).$$

Therefore

$$R(z) = (\rho^*)^{N(r)} z^{N(r)}.\left\{a_0 + a_{p_1}\frac{1}{(\rho^*)^{p_1} z^{p_1}} + \cdots + a_{p_m}\frac{1}{(\rho^*)^{p_m} z^{p_m}} + a_{N(r)}\frac{1}{(\rho^*)^{N(r)} z^{N(r)}}\right\}$$
$$= a_0(\rho^*)^{N(r)} z^{N(r)} + a_{p_1}(\rho^*)^{N(r)-p_1} z^{N(r)-p_1} + \ldots + a_{p_m}(\rho^*)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}.$$

Now

$$|R(z)| \geq |a_0(\rho^*)^{N(r)} z^{N(r)}| - |a_{p_1}(\rho^*)^{N(r)-p_1} z^{N(r)-p_1} + \ldots + a_{p_m}(\rho^*)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}|. \tag{9}$$

Also

$$|a_{p_1}(\rho^*)^{N(r)-p_1} z^{N(r)-p_1} + \ldots + a_{p_m}(\rho^*)^{N(r)-p_m} z^{N(r)-p_m} + a_{N(r)}|$$
$$\leq |a_{p_1}(\rho^*)^{N(r)-p_1} z^{N(r)-p_1}| + \ldots + |a_{p_m}(\rho^*)^{N(r)-p_m} z^{N(r)-p_m}| + |a_{N(r)}|$$
$$\leq |a_{p_1}|(\rho^*)^{N(r)-p_1}|z|^{N(r)-p_1} + \ldots + |a_{p_m}|(\rho^*)^{N(r)-p_m}|z|^{N(r)-p_m} + |a_{N(r)}|$$
$$\leq |a_{p_1}|(\rho^*)^{N(r)-p_1}\left(|z|^{N(r)-p_1} + \cdots + |z|^{N(r)-p_m} + 1\right). \tag{10}$$

Using (10) we get from (9) that for $|z| \neq 0$

$$|R(z)| \geq |a_0|(\rho^*)^{N(r)}|z|^{N(r)} - |a_{p_1}|(\rho^*)^{N(r)-p_1}\left(|z|^{N(r)-p_1} + \cdots + |z| + 1\right)$$
$$= |a_0|(\rho^*)^{N(r)}|z|^{N(r)} - |a_{p_1}|(\rho^*)^{N(r)-p_1}|z|^{N(r)}\left(\frac{1}{|z|^{p_1}} + \cdots + \frac{1}{|z|^{p_m}} + \frac{1}{|z|^{N(r)}}\right)$$

$$> |a_0| (\rho^*)^{N(r)}|z|^{N(r)} - |a_{p_1}| (\rho^*)^{N(r)-p_1}|z|^{N(r)} \left(\frac{1}{|z|^{p_1}} + \cdots\cdots + \frac{1}{|z|^{p_m}} + \frac{1}{|z|^{N(r)}} + \cdots\cdots\right).$$

Therefore for $|z| \neq 0$,

$$|R(z)| > |a_0| (\rho^*)^{N(r)}|z|^{N(r)} - |a_{p_1}| (\rho^*)^{N(r)-p_1}|z|^{N(r)} \left(\sum_{k=1}^{\infty} \frac{1}{|z|^k}\right). \quad (11)$$

Now the geometric series $\sum_{k=1}^{\infty} \frac{1}{|z|^k}$ is convergent for

$$\frac{1}{|z|} < 1$$

$$\text{i.e., for } |z| > 1$$

And converges to

$$\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}} = \frac{1}{|z|-1}.$$

So

$$\sum_{k=1}^{\infty} \frac{1}{|z|^k} = \frac{1}{|z|-1} \quad \text{for } |z| > 1.$$

Therefore for $|z| > 1$,

$$|R(z)| > |a_0| (\rho^*)^{N(r)}|z|^{N(r)} - |a_{p_1}| (\rho^*)^{N(r)-p_1}|z|^{N(r)} \left(\frac{1}{|z|-1}\right)$$
$$= |z|^{N(r)} (\rho^*)^{N(r)-p_1}\left(|a_0| (\rho^*)^{p_1} - \frac{|a_{p_1}|}{|z|-1}\right).$$

i.e., for $|z| > 1$

$$|R(z)| > |z|^{N(r)} (\rho^*)^{N(r)-p_1}\left(|a_0| (\rho^*)^{p_1} - \frac{|a_{p_1}|}{|z|-1}\right).$$

Now

$$R(z) > 0 \text{ if } \left(|a_0| (\rho^*)^{p_1} - \frac{|a_{p_1}|}{|z|-1}\right) \geq 0$$
$$\text{i.e., if } |a_0| (\rho^*)^{p_1} \geq \frac{|a_{p_1}|}{|z|-1}$$
$$\text{i.e., if } |z| - 1 \geq \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}}$$
$$\text{i.e., if } |z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}} > 1.$$

Therefore

$$R(z) > 0 \text{ if } |z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}}.$$

Since $R(z)$ does not vanish in

$$|z| \geq 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}},$$

all the zeros of $R(z)$ lie in

$$|z| < 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}}.$$

Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $a_0 \neq 0$.
Putting $z = \rho^* z_0$ in $R(z)$ we obtain that

$$R\left(\frac{1}{\rho^* z_0}\right) = (\rho^*)^{N(r)}.\left(\frac{1}{\rho^* z_0}\right)^{N(r)} \cdot P(z_0)$$
$$= \left(\frac{1}{z_0}\right)^{N(r)} \cdot 0 = 0.$$

So

$$\left|\frac{1}{\rho^* z_0}\right| < 1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}}$$
$$\text{i.e., } \left|\frac{1}{z_0}\right| < \rho^*\left(1 + \frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}}\right)$$

$$\text{i.e., } |z_0| > \frac{1}{\rho^*\left(1+\frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}}\right)}.$$

As z_0 is an arbitrary zero of $f(z)$, all the zeros of $P(z)$ lie in

$$|z| > \frac{1}{\rho^*\left(1+\frac{|a_{p_1}|}{|a_0| (\rho^*)^{p_1}}\right)}. \quad (12)$$

So from (8) and (12) we may conclude that all the zeros of $P(z)$ lie in the proper ring shaped region

$$\frac{1}{\rho^*\left(1+\frac{|a_{p_1}|}{|a_0|\,(\rho^*)^{p_1}}\right)} < |z| < \frac{1}{\rho^*}\left(1+\frac{|a_0|}{|a_{N(r)}|}(\rho^*)^{N(r)}\right).$$

This proves the theorem.

Corollary 1 In view of Theorem 1 we may conclude that all the zeros of

$$P(z) = a_0 + a_{p_1}z^{p_1} + \ldots\ldots + a_{p_m}z^{p_m} + a_n z^n$$

of degree n with $1 \le p_1 < p_2 \ldots\ldots < p_m \le n-1$, p_i's are integers such that for some $\rho^* > 0$,

$$|a_0| \ge |a_{p_1}| \ge \ldots\ldots \ge |a_{p_m}| \ge |a_n|$$

lie in the ring shaped region

$$\frac{1}{\left(1+\frac{|a_{p_1}|}{|a_0|}\right)} < |z| < \left(1+\frac{|a_0|}{|a_n|}\right)$$

on putting $\rho^* = 1$ in Theorem 1.

Theorem 2 Let $P(z)$ be an entire function having order $\rho = 0$. For sufficiently large r in the disc $|z| \le \log r$, the Taylor's series expansion of $P(z)$ be given by $P(z) = a_0 + a_1 z + \ldots\ldots + a_{N(r)}z^{N(r)}$, $a_0 \ne 0$. Further for some $\rho^* > 0$,

$$|a_0|(\rho^*)^{N(r)} \ge |a_1|(\rho^*)^{N(r)-1} \ge \ldots\ldots \ge |a_{N(r)}|.$$

Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\rho^* t_0'} \le |z| \le \frac{1}{\rho^*}t_0$$

where t_0 and t_0' are the greatest roots of

$$g(t) \equiv |a_{N(r)}|t^{N(r)+1} - \left(|a_{N(r)}| + (\rho^*)^{N(r)}|a_0|\right)t^{N(r)} + (\rho^*)^{N(r)}|a_0| = 0$$

and

$$f(t) \equiv |a_0|\rho^* t^{N(r)+1} - (|a_0|\,\rho^* + |a_1|)t^{N(r)} + |a_1| = 0.$$

Proof. Let

$$P(z) = a_0 + a_1 z + \ldots\ldots + a_{N(r)}z^{N(r)}$$

by applying Lemma 1 and in view of Taylor's series expansion of $P(z)$. Also

$$|a_0|(\rho^*)^{N(r)} \ge |a_1|(\rho^*)^{N(r)-1} \ge \ldots\ldots \ge |a_{N(r)}|.$$

Let us consider

$$Q(z) = (\rho^*)^{N(r)}P\left(\frac{z}{\rho^*}\right)$$

$$= (\rho^*)^{N(r)}\left\{a_0 + a_1\frac{z}{\rho^*} + a_2\frac{z^2}{(\rho^*)^2} + \cdots\cdots + a_{N(r)}\frac{z^{N(r)}}{(\rho^*)^{N(r)}}\right\}$$
$$= a_0(\rho^*)^{N(r)} + a_1(\rho^*)^{N(r)-1}z + \ldots\ldots + a_{N(r)}z^{N(r)}.$$

Now

$$|Q(z)| \ge |a_{N(r)}||z|^{N(r)} - \left|a_0(\rho^*)^{N(r)} + a_1(\rho^*)^{N(r)-1}z + \ldots\ldots + a_{N(r)-1}z^{N(r)-1}\right|.$$

Also applying the condition $|a_0|(\rho^*)^{N(r)} \ge |a_1|(\rho^*)^{N(r)-1} \ge \ldots\ldots \ge |a_{N(r)}|$ we get from above that

$$\left|a_0(\rho^*)^{N(r)} + a_1(\rho^*)^{N(r)-1}z + \ldots\ldots + a_{N(r)-1}z^{N(r)-1}\right|$$
$$\le |a_0|(\rho^*)^{N(r)} + |a_1|(\rho^*)^{N(r)-1}|z| + \ldots\ldots + |a_{N(r)-1}||z|^{N(r)-1}$$
$$\le |a_0|(\rho^*)^{N(r)}\left(1 + |z| + \cdots\cdots + |z|^{N(r)-1}\right)$$
$$= |a_0|(\rho^*)^{N(r)}\cdot\frac{|z|^{N(r)}-1}{|z|-1} \text{ for } |z| \ne 1.$$

Therefore it follows from above that

$$|Q(z)| \ge |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}\cdot\frac{|z|^{N(r)}-1}{|z|-1}.$$

Now

$$|Q(z)| > 0 \text{ if } |a_{N(r)}||z|^{N(r)} - |a_0|(\rho^*)^{N(r)}\cdot\frac{|z|^{N(r)}-1}{|z|-1} > 0$$

i.e., if $|a_{N(r)}||z|^{N(r)} > |a_0|(\rho^*)^{N(r)}\cdot\frac{|z|^{N(r)}-1}{|z|-1}$

i.e., if $|a_{N(r)}||z|^{N(r)}(|z|-1) > |a_0|(\rho^*)^{N(r)}\left(|z|^{N(r)}-1\right)$

i.e., if $|a_{N(r)}||z|^{N(r)+1} - \left(|a_{N(r)}| + |a_0|(\rho^*)^{N(r)}\right)|z|^{N(r)} + |a_0|(\rho^*)^{N(r)} > 0.$

Let us consider

$$g(t) \equiv |a_{N(r)}|t^{N(r)+1} - \left(|a_{N(r)}| + |a_0|(\rho^*)^{N(r)}\right)t^{N(r)} + |a_0|(\rho^*)^{N(r)} = 0. \quad (13)$$

The maximum number of positive roots of (13) is two because maximum number of changes of sign in $g(t) = 0$ is two and if it is less, less by two. Clearly $t = 1$ is a positive root of $g(t) = 0$. Therefore $g(t) = 0$ must have exactly one positive root other than 1. Let the positive root of $g(t)$ be t_1. Let us take $t_0 = \max\{1, t_1\}$. Clearly for $t > t_0$, $g(t) > 0$. If not for some $t_2 > t_0$, $g(t_2) < 0$. Also $g(\infty) > 0$. Therefore $g(t) = 0$ has another positive root in (t_2, ∞) which gives a contradiction .

So for $t > t_0$, $g(t) > 0$. Also $t_0 \ge 1$.Therefore $|Q(z)| > 0$ if $|z| > t_0$. So $Q(z)$ does not vanish in $|z| > t_0$. Hence all the zeros of $Q(z)$ lie in $|z| \le t_0$.

Let $z = z_0$ be a zero of $P(z)$. So $P(z_0) = 0$. Clearly $z_0 \ne 0$ as $a_0 \ne 0$.

Putting $z = \rho^* z_0$ in $Q(z)$ we get that

$$Q(\rho^* z_0) = (\rho^*)^{N(r)} \cdot P(z_0) = (\rho^*)^{N(r)} \cdot 0 = 0.$$

Therefore $z = \rho^* z_0$ is a zero of $Q(z)$. So $|\rho^* z_0| \le t_0$ or $|z_0| \le \frac{1}{\rho^*} t_0 \cdot$ As z_0 is an arbitrary zero of $P(z)$,

all the zeros of $P(z)$ lie in the region $|z| \le \frac{1}{\rho^*} t_0 \cdot$ (14)

In the order to prove the lower bound of Theorem 2 let us consider

$$R(z) = (\rho^*)^{N(r)} z^{N(r)} P\left(\frac{1}{\rho^* z}\right).$$

Then

$$R(z) = (\rho^*)^{N(r)} z^{N(r)} \left(a_0 + \frac{a_1}{\rho^* z} + \cdots + a_{N(r)} \frac{1}{(\rho^*)^{N(r)} z^{N(r)}}\right)$$
$$= a_0 (\rho^*)^{N(r)} z^{N(r)} + a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \ldots\ldots + a_{N(r)}.$$

Now

$$|R(z)| \ge |a_0| (\rho^*)^{N(r)} |z|^{N(r)} - \left|a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \cdots + a_{N(r)}\right|.$$

Also

$$\left|a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \cdots + a_{N(r)}\right| \le |a_1| (\rho^*)^{N(r)-1} |z|^{N(r)-1} + \ldots\ldots + \left|a_{N(r)}\right|.$$

So applying the condition $|a_0| (\rho^*)^{N(r)} \ge |a_1| (\rho^*)^{N(r)-1} \ge \ldots\ldots \ge \left|a_{N(r)}\right|$ we get from above that

$$-\left|a_1 (\rho^*)^{N(r)-1} z^{N(r)-1} + \cdots + a_{N(r)}\right| \ge -|a_1| (\rho^*)^{N(r)-1} |z|^{N(r)-1} - \ldots\ldots - \left|a_{N(r)}\right|$$
$$\ge -|a_1| (\rho^*)^{N(r)-1} \left(|z|^{N(r)-1} + \cdots + 1\right)$$
$$= -|a_1| (\rho^*)^{N(r)-1} \cdot \frac{|z|^{N(r)-1} - 1}{|z| - 1} \text{ for } |z| \ne 1. \quad (15)$$

Using (15) we get for $|z| \ne 1$ that

$$|R(z)| \ge (\rho^*)^{N(r)-1} \left(|a_0| \rho^* |z|^{N(r)} - |a_1| \cdot \frac{|z|^{N(r)-1} - 1}{|z| - 1}\right). \quad (16)$$

Now

$$R(z) > 0 \text{ if } (\rho^*)^{N(r)-1} \left(|a_0| \rho^* |z|^{N(r)} - |a_1| \cdot \frac{|z|^{N(r)-1} - 1}{|z| - 1}\right) > 0$$
$$\text{i.e., if } |a_0| \rho^* |z|^{N(r)} - |a_1| \cdot \frac{|z|^{N(r)-1} - 1}{|z| - 1} > 0$$
$$\text{i.e., if } |a_0| \rho^* |z|^{N(r)} > |a_1| \cdot \frac{|z|^{N(r)-1} - 1}{|z| - 1}$$
$$\text{i.e., if } |a_0| \rho^* |z|^{N(r)} (|z| - 1) > |a_1| \left(|z|^{N(r)-1} - 1\right)$$
$$\text{i.e., if } |a_0| \rho^* |z|^{N(r)+1} - (|a_0| \rho^* + |a_1|) |z|^{N(r)} + |a_1| > 0.$$

Let us consider

$$f(t) \equiv |a_0| \rho^* t^{N(r)+1} - (|a_0| \rho^* + |a_1|) t^{N(r)} + |a_1| = 0.$$

Clearly $f(t) = 0$ has two positive roots, because the number of changes of sign of $f(t)$ is two. If it is less, less by two. Also $t = 1$ is the one of the positive roots of $f(t) = 0$. Let us suppose that $t = t_2$ be the other positive root. Also let $t_0' = \max\{1, t_2\}$ and so $t_0' \ge 1$. Now $t > t_0'$ implies $f(t) > 0$. If not then there exists some $t_3 > t_0'$ such that $f(t_3) < 0$. Also $f(\infty) > 0$. Therefore there exists another positive root in (t_3, ∞) which is a contradiction.

So $|R(z)| > 0$ if $|z| > t_0'$. Thus $R(z)$ does not vanish in $|z| > t_0'$. In otherwords all the zeros of $R(z)$ lie in $|z| \le t_0'$.

Let $z = z_0$ be any zero of $P(z)$. So $P(z_0) = 0$. Clearly $z_0 \ne 0$ as $a_0 \ne 0$.

Putting $z = \frac{1}{\rho^* z_0}$ in $R(z)$ we get that

$$R\left(\frac{1}{\rho^* z_0}\right) = (\rho^*)^{N(r)} \cdot \left(\frac{1}{\rho^* z_0}\right)^{N(r)} \cdot P(z_0) = \left(\frac{1}{z_0}\right)^{N(r)} \cdot 0 = 0.$$

Therefore $\frac{1}{\rho^* z_0}$ is a root of $R(z)$. So $\left|\frac{1}{\rho^* z_0}\right| \le t_0'$ implies $|z_0| \ge \frac{1}{\rho^* t_0'} \cdot$ As z_0 is an arbitrary zero of $P(z) = 0$,

all the zeros of $P(z)$ lie in $|z| \ge \frac{1}{\rho^* t_0'} \cdot$ (17)

From (14) and (17) we have all the zeros of $P(z)$ lie in the ring shaped region given by

$$\frac{1}{\rho^* t_0'} \le |z| \le \frac{1}{\rho^*} t_0$$

where t_0 and t_0' are the greatest positive roots of $g(t) = 0$ and $f(t) = 0$ respectively.

This proves the theorem.

Corollary 2 From Theorem 2 we can easily conclude that all the zeros of

$$P(z) = a_0 + a_1 z + \ldots\ldots + a_n z^n$$

of degree n with property $|a_0| \ge |a_1| \ge \ldots\ldots \ge |a_n|$ lie in the ring shaped region

$$\frac{1}{t_0'} \le |z| \le t_0$$

where t_0 and t_0' are the greatest positive roots of

$$g(t) \equiv |a_n| t^{n+1} - (|a_n| + |a_0|) t^n + |a_0| = 0$$

and

$$f(t) \equiv |a_0|t^{n+1} - (|a_0| + |a_1|)t^n + |a_1| = 0$$

respectively by putting $\rho^* = 1$.

Remark 1 The limit of Theorem 2 is attained by $P(z) = a^2z^2 - az - 1,\ a > 0$. Here $P(z) = a^2z^2 - az - 1$, $a_0 = -1$, $a_1 = -a$, $a_2 = a^2$. Therefore $|a_0| = 1$, $|a_1| = a$, $|a_2| = a^2$. Let $\rho^* = a$. So $|a_0|(\rho^*)^2 \geq |a_1|\,\rho^* \geq |a_2|$ holds. Hence

$$g(t) \equiv |a_2|t^3 - (|a_2| + a^2|a_0|)t^2 + |a_0|a^2 = 0$$
$$\text{i.e., } a^2(t^3 - 2t^2 + 1) = 0.$$

Now $g(t) = 0$ has two positive roots which are $t_1 = 1$ and $t_2 = \frac{\sqrt{5}+1}{2}$. So $t_0 = \max\,(t_1, t_2) = \frac{\sqrt{5}+1}{2}$.

$$f(t) \equiv |a_0|\rho^* t^3 - (|a_0|\,\rho^* + |a_1|)t^2 + |a_1| = 0$$
$$\text{i.e., } at^3 - (1.a + a)t^2 + a = 0$$
$$\text{i.e., } a(t^3 - 2t^2 + 1) = 0$$
$$\text{i.e., } t = 1 \text{ and } t = \frac{\sqrt{5}+1}{2}.$$

Again

$$t_0' = \max\ (\text{positive roots of } f(t) = 0)$$
$$= \frac{\sqrt{5}+1}{2}.$$

Hence by Theorem 2, all the zeros lie in

$$\frac{1}{\rho^* t_0'} \leq |z| \leq \frac{1}{\rho^*} t_0$$
$$\text{i.e., } \frac{1}{a\,\frac{\sqrt{5}+1}{2}} \leq |z| \leq \frac{1}{a}\frac{\sqrt{5}+1}{2}$$
$$\text{i.e., } \frac{\sqrt{5}-1}{2a} \leq |z| \leq \frac{\sqrt{5}+1}{2a}.$$

Now

$$P(z) = 0$$
$$\text{i.e., } a^2z^2 - az - 1 = 0$$
$$\text{i.e., } z = \frac{1 \pm \sqrt{5}}{2a}.$$

Let

$$z_1 = \frac{1+\sqrt{5}}{2a} \text{ and } z_2 = \frac{1-\sqrt{5}}{2a}.$$

Clearly z_1 lie on the upper bound and z_2 lie on the lower bound of the boundary. Also here the order $\rho = 0$ because $M(r) = |a^2|r^2 = a^2r^2$ for large r in the circle $|z| = r$. Therefore

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \limsup_{r\to\infty} \frac{\log\log a^2r^2}{\log r}$$
$$= \limsup_{r\to\infty} \frac{\frac{1}{\log\ a^2r^2}\cdot\frac{1}{a^2r^2}\cdot 2a^2r}{\frac{1}{r}} = \limsup_{r\to\infty} \frac{2}{\log\ a^2r^2} = 0.$$

Also $\rho^* = 2$ and $N(r) = 2 \leq (\log r)^{2+\varepsilon}$ for $\varepsilon > 0$ and sufficiently large r in $|z| \leq \log r$ and $a_n = 0$ for $n \ngtr N(r)$.

Corollary 3 Under the conditions of Theorem 2 and

$$P(z) = a_0 + a_{p_1}z^{p_1} + \ldots\ldots + a_{p_m}z^{p_m} + a_{N(r)}z^{N(r)}$$

with

$$1 \leq p_1 \leq p_2 \ldots\ldots \leq p_m \leq N(r) - 1,$$

where p_i's are integers $a_0, a_{p_1}, \ldots\ldots, a_{N(r)}$ are non vanishing coefficients with

$$|a_0|(\rho^*)^{N(r)} \geq |a_{p_1}|(\rho^*)^{N(r)-p_1} \geq \ldots\ldots \geq |a_{p_m}|(\rho^*)^{N(r)-p_m} \geq |a_{N(r)}|$$

then we can show that all the zeros of $P(z)$ lie in

$$\frac{1}{\rho^* t_0'} \leq |z| \leq \frac{1}{\rho^*} t_0$$

where t_0 and t_0' are the greatest positive roots of

$$g(t) \equiv |a_{N(r)}|t^{N(r)+1} - \left(|a_{N(r)}| + |a_0|\,(\rho^*)^{N(r)}\right)t^{N(r)} + |a_0|\,(\rho^*)^{N(r)} = 0$$

and

$$f(t) \equiv |a_0|\,(\rho^*)^{p_1}\,t^{N(r)+1} - \left(|a_0|\,(\rho^*)^{p_1} + |a_{p_1}|\right)t^{N(r)} - |a_{p_1}| = 0 \text{ respectively.}$$

Corollary 4 If we put $\rho^* = 1$ in Corollary 3 then all the zeros of

$$P(z) = a_0 + a_{p_1}z^{p_1} + \ldots\ldots + a_{p_m}z^{p_m} + a_nz^n$$

lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where t_0 and t_0' are the greatest positive roots of

$$g(t) \equiv |a_n|t^{n+1} - (|a_n| + |a_0|)t^n + |a_0| = 0$$

and

$$f(t) \equiv |a_0|\,t^{n+1} - \left(|a_0| + |a_{p_1}|\right)t^n - |a_{p_1}| = 0 \text{ respectively}$$

provided

$$|a_0| \geq |a_{p_1}| \geq \ldots\ldots\geq |a_{p_m}| \geq |a_n| \cdot$$

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