On the Exponential Diophantine Equations $x^x y^{y^n} = z^{z^n}$ and $x^{x^n} y^{y^m} = z^{z^n}$

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ABSTRACT: In this paper,two different forms of exponential Diophantine equations namely $x^x y^{y^n} = z^{z^n}$ and $x^{x^n} y^{y^m} = z^{z^n}$ are considered and analysed for finding positive integer solutions on each of the above two equations.some numerical examples are presented in each case.

Keywords: *Exponential Diophantine Equation,integral solutions.*
Mathematics subject classification number: *11D61*

I. INTRODUCTION

The exponential diophantine equation $a^x + b^y = c^z$ in positive integers x, y, z has been studied by number of authors [1-5].In [6-12] the existence and the processes of determining some positive integer solutions to a few special cases of an exponential diophantine equation are studied. In this paper, two different representations I and II of the exponential diophantine equations namely $x^x y^{y^n} = z^{z^n}$ and $x^{x^n} y^{y^m} = z^{z^n}$ are studied with some numerical examples.

II. METHOD OF ANALYSIS

Representation I

The exponential diophantine equation with three unknowns to be solved for its non-zero distinct integral solutions is

$$x^x y^{y^n} = z^{z^n} \qquad (1)$$

where n is a natural numbers

Introducing the transformations

$$x = uz^n, y = v^{\frac{1}{n}} z \qquad (2)$$

in (1) ,it becomes

$$z = u^{\frac{u}{1-nu-v}} v^{\frac{\frac{v}{n}}{1-nu-v}} \qquad (3)$$

Taking

$$\frac{u}{1-nu-v} = -n_1 , \quad \frac{v}{n(1-nu-v)} = -n_2 \qquad (4)$$

and solving the above two equations, we have

$$u = \frac{n_1}{nn_1 + nn_2 - 1} , \ v = \frac{nn_2}{nn_1 + nn_2 - 1} \qquad (5)$$

Substituting (5) in(3) and (2),the corresponding solutions of (1) are

$$\left.\begin{aligned} x &= \left(\frac{nn_1 + nn_2 - 1}{n_1}\right)^{nn_1 - 1} \left(\frac{nn_1 + nn_2 - 1}{nn_2}\right)^{nn_2} \\ y &= \left(\frac{nn_1 + nn_2 - 1}{n_1}\right)^{n_1} \left(\frac{nn_1 + nn_2 - 1}{nn_2}\right)^{n_2 - \frac{1}{n}} \\ z &= \left(\frac{nn_1 + nn_2 - 1}{n_1}\right)^{n_1} \left(\frac{nn_1 + nn_2 - 1}{nn_2}\right)^{n_2} \end{aligned}\right\} \ldots\ldots\ldots\ldots(6)$$

The numbers n_1 and n_2 can be chosen such that the solutions (6) be natural numbers.

Now taking

$n_1 = n^{\alpha n-1}$, $\alpha > 0$; $n_2 = \frac{1}{n}$, n>0 in (5) ,the non-zero integral solutions of (2) are found to be

$$x = n^{n^{\alpha n}} n^{\alpha n-1}$$

$$y = n^{n^{\alpha n-1}}$$

$$z = n^{n^{\alpha n-1}} n^{\alpha}$$

Numerical Examples

(α, n)	x	y	z
(1,2)	32	4	8
(2,3)	3^{734}	3^{243}	3^{245}
(1,3)	3^{29}	3^{9}	3^{10}
(2,2)	2^{19}	2^{8}	2^{10}

Representation II

The exponential Diophantine equation with three unknowns to be solved for its non-zero distinct integral solutions is

$$x^{x^n} y^{y^m} = z^{z^n} \quad (7)$$

where m,n are natural numbers

Considering the transformations

$$x = u^{\frac{1}{n}} z, \quad y = v^{\frac{1}{m}} z^{\frac{n}{m}} \quad (8)$$

in (7), it can be written as

$$z = u^{\frac{\frac{u}{n}}{1-u-\frac{nv}{m}}} v^{\frac{\frac{v}{m}}{1-u-\frac{nv}{m}}} \quad (9)$$

Assuming

$$\frac{\frac{u}{n}}{1-u-\frac{nv}{m}} = -n_1, \quad \frac{\frac{v}{m}}{1-u-\frac{nv}{m}} = -n_2 \quad (10)$$

and solving the above two equations, we have

$$u = \frac{nn_1}{nn_1 + nn_2 - 1}, \quad v = \frac{mn_2}{nn_1 + nn_2 - 1} \quad (11)$$

Substituting (11) in(9) and (8),the corresponding solutions of (6) are

$$\left.\begin{aligned} x &= \left(\frac{nn_1 + nn_2 - 1}{nn_1}\right)^{\frac{nn_1 - 1}{n}} \left(\frac{nn_1 + nn_2 - 1}{mn_2}\right)^{n_2} \\ y &= \left(\frac{nn_1 + nn_2 - 1}{n_1}\right)^{\frac{nn_1}{m}} \left(\frac{nn_1 + nn_2 - 1}{mn_2}\right)^{\frac{nn_2 - 1}{m}} \\ z &= \left(\frac{nn_1 + nn_2 - 1}{nn_1}\right)^{n_1} \left(\frac{nn_1 + nn_2 - 1}{mn_2}\right)^{n_2} \end{aligned}\right\} \ldots\ldots\ldots(12)$$

The numbers n_1 and n_2 can be chosen such that the solutions (11) be natural numbers.
For illustration, Choosing

$n = m\beta^m, n_2 = \alpha^{mn} n^{mn-1}, n_1 = \frac{1}{n}$, n>0 in (11) ,the non-zero integral solutions of (6) are represented by

$$x = \beta^{m\alpha^{mn} n^{mn-1}}$$

$$y = (\alpha n)^n \beta^{(\alpha n)^{mn} - 1}$$

$$z = (\alpha n)^m \beta^{m\alpha^{mn} n^{mn-1}}$$

Numerical examples:

(m,n)	(α,β)	x	y	z
(1,2)	(1,2)	2^2	2^5	2^3
(1,3)	(2,18)	$3^{2.18^{35}}$	$18^{18} 3^{18^{36}-1}$	$18^2 3^{2.18^{35}}$

III. CONCLUSION

To conclude,one may search for other pattern of integer solutions to the above exponential diophantine equations.

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