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Hopf-Bifurcation Ina Two Dimensional Nonlinear Differential Equation

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ABSTRACT:*In this paper we have investigated the stability nature of Hopf bifurcation in a two dimensional nonlinear differential equation.Interestingly, our considered model exhibits both supercritical and subcritical Hopf bifurcation for certain parameter values which marks the stability and instability of limit cycles created or destroyed in Hopf bifurcations respectively. We have used the Center manifold theorem and the technique of Normal forms in our investigation.*

Keywords: *Hopf bifurcation, supercritical Hopf bifurcation, subcritical Hopf bifurcation, Centre manifold, normal form.*

I. INTRODUCTION

From historical point of view, search on the existence of periodic solution plays a fundamental role in the development of qualitative study of dynamical systems. There are many methods for locating the periodic solutions, for example the Poincare return map, Melnikov integral, bifurcation theory etc. In this paper we have usedCenter manifold theorem, Normal forms and Hopf bifurcation theorem to study the behaviour of the limit cycle.

The term Hopf bifurcation (also sometimes called Poincare Andronov-Hopf bifurcation) refers to the local birth or death of a periodic solution (self-excited oscillation) from an equilibrium as a parameter crosses a critical value. In a differential equation a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearized flow at a fixed point becomes purely imaginary.The uniqueness of such bifurcations lies in two aspects: unlike other common types of bifurcations (viz., pitchfork, saddle-node or transcritical) Hopf bifurcation cannot occur in one dimension. The minimum dimensionality has to be two. The other aspect is that Hopf bifurcation deals with birth or death of a limit cycle as and when it emanates from or shrinks onto a fixed point, the focus. Recently, Hopf bifurcation of some famous chaotic systems has been investigated and it is becoming one of the most active topics in the field of chaotic systems.

Hopf bifurcation has played a pivotal role in the development of the theory of dynamical systems in different dimensions. Following Hopf'soriginal work [1942], Hopf and generalized Hopf bifurcationshave beenextensively studied by many researchers[3,10,15,19,20,21,24,29,32,33,37].

As is well known, Hopf bifurcation gives rise to limit cycles, which are typical oscillatory behaviors of many nonlinear systems in physical, social, economic, biological, and chemical fields. These oscillatory behaviorscan be beneficial in practical applications, such as in mixing, monitoring, and fault diagnosis in electromechanical systems[11,15,25]. Also, the properties of limit cycles are very useful in modern control engineering, such as auto tuning of PID controller [39] and process identification [36]. Early efforts in Hopf bifurcation control [38] focused on delaying the onset of this bifurcation[1] or stabilizing an existing bifurcation [40].

II. CENTRE MANIFOLD THEOREM AND ITS ROLE IN HOPF BIFURCATION

The center manifold theorem in finite dimensions can be traced to the work of Pliss [28], Sositaisvili [31] and Kelley [22]. Additional valuable references are Guckenheimer and Holmes [15], Hassard, Kazarinoff, and Wan [16], Marsden and McCracken [24], Carr [7], Henry [18], Sijbrand [30], Wiggins [37] andPerko[27].

The center manifold theorem is a model reduction technique for determining the local asymptotic stability of an equilibrium of a dynamical system when its linear part is not hyperbolic. The overall system is asymptotically stable if and only if the Center manifold dynamics is asymptotically stable. This allows for a substantial reduction in the dimension of the system whose asymptotic stability must be checked. In fact, the center manifold theorem is used to reduce the system from N dimensions to 2 dimensions[17]. Moreover, the

Centermanifold and its dynamics need not be computed exactly; frequently, a low degree approximation is sufficientto determine its stability [4].
We consider a nonlinear system

$$\dot{x} = f(x), x \in R^n \qquad (1)$$

Suppose that the system (1) has an equilibrium point x_0 at the parameter value $\mu = \mu_0$ such that $f(x_0) = 0$. In order to study the behaviour of the system near x_0 we first linearise the system (1)atx_0. The linearised system is

$$\dot{x} = Ax, x_0 \in R^n \qquad (2)$$

Where $A = Df(x_0)$ is the Jacobian matrix of f of order $n \times n$. The system has invariant subspaces E^s, E^u, E^c, corresponding to the span of the generalised eigenvectors, which in turn correspond to eigenvalues having negative real part, positive real part and zero real part respectively.The subspaces are so named because orbits starting in E^s decay to zero as t tends to ∞, orbits starting in E^u become unbounded as t tends to ∞ and orbits starting in E^c neither grow nor decay exponentially as t tends to ∞. Theoretically, it is already established that if we suppose $E^u = \phi$, then any orbit will rapidly decay to E^c. Thus, if we are interested in long term behaviour (i.e. stability) we need only to investigate the system restricted to E^c.

The Hartman-GrobmanTheorem[21] says that in a neighbourhood of a hyperbolic critical point x_0 , the nonlinear system (1) is topologically conjugate to the linear system (2), in a neighbourhood of the origin. The Hartman-Grobman theorem therefore completely solves the problem of determining the stability and qualitative behaviour in a neighbourhood of a hyperbolic critical point.

In case of non-hyperbolic critical point, the Hartman-Grobman Theorem is not applicable and its role is played by the centre manifold thorem. The center manifold theorem shows that the qualitative behaviour in a neighbourhood of a non-hyperbolic critical point x_0 of the nonlinear system (1) with $x \in R^n$ is determined by its behaviour on the center manifold near x_0. Since the center manifold is generally of smaller dimension than the system (1), this simplifies the problem of determining the stability and qualitative behaviour of the flow near a non-hyperbolic critical point of (1).

III. CENTER MANIFOLD THEOREM FOR FLOWS

The statement of the Center Manifold Theorem for Flows is as follows :
Let f be a C^r vector field on R^n vanishing at the origin ($f(0) = 0$) and let $A = Df(0)$. Divide the spectrum of A into three parts, $\sigma_s, \sigma_c, \sigma_u$ with

$$\text{Re}\,\lambda \begin{cases} < 0 & if \lambda \in \sigma_s \\ = 0 & if \lambda \in \sigma_c \\ > 0 & if \lambda \in \sigma_u \end{cases}$$

Let the (generalised) eigenspaces of $\sigma_s, \sigma_c and \sigma_u$be $E^s, E^c and E^u$, respectively. Then there exist C^r stable and unstable invariant manifolds W^s and W^u tangent to $E^s$and E^u at 0 and a C^{r-1} center manifold W^c tangent to E^c at 0. The manifolds W^s, W^u, W^c are all invariant for the flow of f. The stable and unstable manifolds are unique, but W^c need not be [7,15].
The system (1) can be written in diagonal form

$$\begin{aligned} \dot{x} &= A^c x + f_1(x, y, z) \\ \dot{y} &= A^s y + f_2(x, y, z) \\ \dot{z} &= A^u z + f_3(x, y, z) \end{aligned} \qquad (3)$$

with

$$\begin{aligned} &f_1(0, 0, 0) = 0, \quad && Df_1(0,0,0) = 0 \\ &f_2(0,0,0) = 0, \quad && Df_2(0,0,0) = 0 \\ &f_3(0,0,0) = 0, \quad && Df_3(0,0,0) = 0 \end{aligned}$$

Wheref_1, f_2, f_3 are some $C^r, (r \geq 2)$ in some neighbourhood of the origin,$A^c, A^s$and A^u on the blocks are in the canonical form whose diagonals contain the eigenvalues with $Re\lambda = 0$, $Re\lambda < 0$ and $Re\lambda > 0$, respectively, $(x, y, z) \in R^c \times R^s \times R^u$, $c = \dim E^c$ since the system (3) has c eigenvalues with zero real part, $s = dimE^s$ and $u = DimE^u$, f_1, f_2 and f_3 vanish along with their first partial derivatives at the origin.
If we assume that the unstable manifold is empty, then (3) becomes

$$\begin{aligned} \dot{x} &= A^c x + f_1(x, y) \\ \dot{y} &= A^s y + f_2(x, y) \end{aligned} \qquad (4)$$

where

$$\begin{aligned} &f_1(0,0) = 0, \quad && Df_1(0,0) = 0, \\ &f_2(0,0) = 0 \quad && Df_2(0,0) = 0, \end{aligned}$$

$A^c$is a $c \times c$ matrix having eigenvalues with zero real parts, A^s is an $s \times s$ matrix having eigenvalues with negative real parts, and f_1 and f_2 are C^r functions ($r \geq 2$.)
The following theorems established by Carr [7] in fact forms the basis of our discussion on the role of Centre manifold theorem in Hopf bifurcation and so we have stated them below :

Theorem3.1:
There exists a C^r center manifold,

$$W^c = \{(x,y)|y = h(x), |x| < \delta,\ h(0) = 0, Dh(0) = 0\},$$

forδ sufficiently small,for(4) such that the dynamics of (4) restricted to the center manifold is given by the c-dimensional vector field

$$\dot{u} = A^c u + f_1(u, h(u)) \qquad (5)$$

Theorem3.2:
(i) Suppose the zero solution of (5) is stable (asymptotically stable)(unstable) then the zero solution of (4) is also stable (asymptotically stable) (unstable).
(ii) Suppose the zero solution of (5) is stable. Then if $(x(t), y(t))$ is a solution of (4) with $\big(x(0), y(0)\big)$ sufficiently small, there is a solution $u(t)$ of (5) such that as $t \to \infty$

$$x(t) = u(t) + O(e^{-\gamma t})$$
$$y(t) = h(u(t)) + O(e^{-\gamma t})$$

where$\gamma > 0$ is a constant.
From the theorem 2.2 it is clear that the dynamics of (5) near $u = 0$ determine the dynamics of (4) near $(x, y) = (0,0)$.
To calculate $h(x)$ substitute $y = h(x)$ in the second component of (3) and using the chain rule, we obtain

$$N\big(h(x)\big) = Dh(x)\big[A^c x + f_1\big(x, h(x)\big)\big] - Ch(x) - g\big(x, h(x)\big) = 0 \qquad (6)$$

with boundary conditions $h(0) = Dh(0) = 0$. This differential equation for h cannot be sloved exactly in most cases, but its solution can be approximated arbitrarily closely as a Taylor series at $x = 0$.

Theorem 3.3 :
If a function $\phi(x)$, with $\phi(0) = D\phi(0) = 0$, can be found such that $N\big(\phi(x)\big) = O(|x|^p)$ for some $p > 1$ as $|x| \to 0$ then it follows that

$$h(x) = \phi(x) + O(|x|^p) \text{as} |x| \to 0$$

This theorem allows us to compute the center manifold to any desired degree of accuracyby solving (6) to the same degree of accuracy.
In the discussion above we have assumed the unstable manifold is empty at the bifurcation point. If we include the unstable manifold then we must deal with the system (3). In this case (x,y, z) = (0,0,0) is unstable due to the existence of a u-dimensional unstable manifold. However, much of the center manifold theory still applies, in particular Theorem 2.1 concerning existence, with the center manifold being locally represented by

$$W^c(0) = \{(x, y, z) \in R^c \times R^s \times R^u | y = h_1(x), z = h_2(x), h_i(0) = 0, Dh_i(0) = 0, i = 1,2\}$$

for$x$ sufficiently small. The vector field restricted to the center manifold isgiven by

$$\dot{u} = Au + f(x, h_1(u), h_2(u))$$

IV. HOPF BIFURCATION AND NORMAL FORM

One of the basictools in the study of dynamical behavior of a system governed by nonlinear differential equations near a bifurcation point is the theory of normal forms. Normal form theory has been widely used in the study of nonlinear vector fields in order to simplify the analysis of the original system [5, 8,9, 14,15,23, 26,47].

Several efficient methodologies for computing normal forms have been developed in the past decade [2,12, 13, 34,41, 42,43, 44,45,46].

The method of normal forms can be traced back to the Ph.D thesis of Poincare. The books by Van der Meer [35] and Bryuno [6] give valuable historical background.

The basic idea of normal form theory is to employ successive, near identity nonlinear transformations to eliminate the so called non-resonant nonlinear terms, and retaining the terms which can not be eliminated (called resonant terms) to form the normal form and which is sufficient for the study of qualitative behavior of the original system.

The Local Center Manifold Theorem in the previous section showed us that, in a neighborhood of a nonhyperbolic critical point, determining the qualitative behavior of (1) could be reduced to the problem of determining the qualitative behavior of the nonlinear system

$$\dot{x} = A^c x + F(x) \qquad (7)$$

on the center manifold. Since the dimension of the center manifold is typically less than n, this simplifies the problem of determining the qualitative behavior of the system (1) near a nonhyperbolic critical point. However, analyzing this system still may be a difficult task. The normal form theory allows us to simplify the nonlinear part,F(x), of (7) in order to make this task as easy as possible. This is accomplished by making a nonlinear, analytic transformation (called near identity transformation) of coordinates of the form

$x = y + h(y)$, where $h(y) = O(|y|^2)$ as $|y| \to 0$. (8)

Suppose that $Df_\mu(x_0)$ has two purely imaginary eigenvalues with the remaining $n-2$ eigenvalues having nonzero real parts. We know that since the fixed point is not hyperbolic, the orbit structure of the linearised vector field near $(x, \mu) = (x_0, \mu_0)$ may reveal little (and, possibly, even incorrect) information concerning the nature of the orbit structure of the nonlinear vector field (1) near $(x, \mu) = (x_0, \mu_0)$. But by the center manifold theorem, we know that the orbit structure near $(x, \mu) = (x_0, \mu_0)$ is determined by the vector field (1) restricted to the center manifold.

On the center manifold the vector field (1) has the following form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} Re\lambda(\mu) & -Im\lambda(\mu) \\ Im\lambda(\mu) & Re\lambda(\mu) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x,y,z) \\ f^2(x,y,z) \end{pmatrix} \qquad (x,y,\mu) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \quad (9)$$

Where f^1 and f^2 are nonlinear in x and y and $\lambda(\mu)$, $\overline{\lambda(\mu)}$ are the eigenvalues of the vector field linearized about the fixed point at the origin.

Now, if we denote the eigenvalue

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu), \quad (10)$$

Then for our assumption of non hyperbolic nature we have

$$\alpha(0) = 0, \quad \omega(0) \neq 0 \quad (11)$$

Now we transform the equation (9) to the normal form. The normal form is found to be

$$\dot{x} = \alpha(\mu)x - \omega(\mu)y + (a(\mu)x - b(\mu)y)(x^2 + y^2) + O(|x|^5, |y|^5)$$
$$\dot{y} = \omega(\mu)x + \alpha(\mu)y + (b(\mu)x + a(\mu)y)(x^2 + y^2) + O(|x|^5, |y|^5) \quad (12)$$

In polar coordinates the equation (12) can be written as

$$\dot{r} = \alpha(\mu)r + a(\mu)r^3 + O(r^5),$$
$$\dot{\theta} = \omega(\mu) + b(\mu)r^2 + O(r^4) \quad (13)$$

Since we are interested in the dynamics near $\mu = 0$, therefore expanding in Taylors series the coefficients in (13) about $\mu = 0$, the equation (13) becomes

$$\dot{r} = \alpha'(0)\mu r + a(0)r^3 + O(\mu^2 r, \mu r^3, r^5),$$
$$\dot{\theta} = \omega(0) + \omega'(0)\mu + b(0)r^2 + O(\mu^2, \mu r^2, r^4) \quad (14)$$

where " $'$ " denotes differentiation with respect to μ and we have used the fact that $\alpha(0) = 0$.

Our goal is to understand the dynamics of (14) for small r and μ. This is accomplished in two steps. In the first step, weneglect the higher order terms of (14) to get a "truncated" normal form and in the second step we study the dynamics exhibited by the truncated normal form as it is well known that the dynamics exhibited by the truncated normal form are qualitatively unchanged when one considers the influence of the previously neglected higher order terms [37].

Now, neglecting the higher order terms in (14) gives

$$\dot{r} = d\mu r + ar^3,$$
$$\dot{\theta} = \omega + c\mu + br^2 \quad (15)$$

where for ease of notation, we have

$\alpha'(0) \equiv d, a(0) \equiv a, \quad \omega(0) \equiv \omega, \quad \omega'(0) \equiv c, \quad b(0) \equiv b$

For $-\infty < \frac{\mu d}{a} < 0$ and μ sufficiently small, the solution of (15) is given by

$$\big(r(t), \theta(t)\big) = \left(\sqrt{-\frac{\mu d}{a}}, \quad \left[\omega + \left(c - \frac{bd}{a}\right)\mu\right] t + \theta_0 \right)$$

which is a periodic orbit for (15) and the periodic orbit is asymptotically stable for $a < 0$ and unstable for $a > 0$ [15, 37].

In practice, a is straight forward to calculate. It can be calculated simply by keeping track of the coefficients carefully in the normal form transformation in terms of the original vector field. The expression of a for a two dimensional system of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

with $f(0) = 0 = g(0) = 0$, is found to be [15, 16, 24]

$$a = \frac{1}{16}\left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right) + \frac{1}{16\omega}\left(f_{xy}\left(f_{xx} + f_{yy}\right) - g_{xy}\left(g_{xx} + g_{yy}\right) - f_{xx}g_{xx} + f_{yy}g_{yy}\right)$$

In fact, all the above discussions gets converged in the famous Hopf Bifurcation Theorem which states as follows :

Let the eigenvalues of the linearized system of $\dot{x} = f_\mu(x)$, $x \in R^n$, $\mu \in R$ about the equilibrium point be given by $\lambda(\mu), \bar{\lambda}(\mu) = \alpha(\mu) \pm i\beta(\mu)$. Suppose further that for a certain value of μ say $\mu = \mu_0$, the following conditions are satisfied:

1. $\alpha(0) = 0, \beta(0) = \omega \neq 0$
2. $\left.\frac{d\alpha(\mu)}{d\mu}\right|_{\mu=\mu_0} = d \neq 0$

(transversality condition: the eigenvalues cross the imaginary axis with nonzero speed)

3. $a \neq 0$, where

$$a = \frac{1}{16}\left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right) + \frac{1}{16\omega}\left(f_{xy}\left(f_{xx} + f_{yy}\right) - g_{xy}\left(g_{xx} + g_{yy}\right) - f_{xx}g_{xx} + f_{yy}g_{yy}\right)$$

where f_{xy} denotes $\left.\frac{\partial^2 f_\mu}{\partial x \partial y}\right|_{\mu=\mu_0}(x_0, y_0)$, etc.

(genericity condition)

then there is a unique three-dimensional center manifold passing through (x_0, μ_0) in $R^n \times R$ and a smooth system of coordinates (preserving the planes $\mu = const.$) for which the Taylor expansion of degree 3 on the center manifold is given by (12). If $a \neq 0$, there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of $\lambda(\mu_0), \bar{\lambda}(\mu_0)$agreeing to second order with the paraboloid$\mu = -(a/d)(x^2 + y^2)$. If $a < 0$, then these periodic solutions are stable limit cycles, while if $a > 0$, the periodic solutions are repelling (unstable limit cycles).

V. DISCUSSION ON THE BASIS OF OUR CONSIDERED MODEL

We discuss the following modeltaken from [15].

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 + \mu_2 y + \mu_3 x^2 + \mu_4 xy \end{aligned} \qquad (16)$$

In our analysis, we fix μ_3 and μ_4 for simplicity. It is seen that, with suitable rescaling and letting $(x, y) \to (-x, -y)$, for any $\mu_3, \mu_4 \neq 0$ the possible cases can be reduced to two: $\mu_3 = 1$ and $\mu_4 = \pm 1$.

Case 1: $\mu_3 = 1$, $\mu_4 = -1$(Super-critical Hopf bifurcation)

In this case the system (2) becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 + \mu_2 y + x^2 - xy \end{aligned} \qquad (17)$$

The fixed points of (17) are given by

$$(x, y) = \left(\pm\sqrt{-\mu_1}, 0\right)$$

which exists for $\mu_1 \leq 0$. We denote both the fixed points as $P_1 = (x_+, 0) = (+\sqrt{-\mu_1}, 0)$and $P_2 = (x_-, 0) = (-\sqrt{-\mu_1}, 0)$, where $x_\pm = \pm\sqrt{-\mu_1}$.

The Jacobian matrix J for the linearized system is

$$J = \begin{pmatrix} 0 & 1 \\ 2x - y & \mu_2 - x \end{pmatrix}$$

The Jacobian matrix J at the fixed point $P_1 = (\sqrt{-\mu_1}, 0)$ is

$$J = \begin{pmatrix} 0 & 1 \\ 2\sqrt{-\mu_1} & \mu_2 - \sqrt{-\mu_1} \end{pmatrix}$$

The eigenvalues are given by

$$\lambda = \frac{\mu_2 - \sqrt{-\mu_1} \pm \sqrt{(\mu_2 - \sqrt{-\mu_1})^2 + 8\sqrt{-\mu_1}}}{2}$$

Let $\lambda_1 = \frac{\mu_2 - \sqrt{-\mu_1} + \sqrt{(\mu_2 - \sqrt{-\mu_1})^2 + 8\sqrt{-\mu_1}}}{2}$ & $\lambda_2 = \frac{\mu_2 - \sqrt{-\mu_1} - \sqrt{(\mu_2 - \sqrt{-\mu_1})^2 + 8\sqrt{-\mu_1}}}{2}$

We observe that$\lambda_1 > 0$ and $\lambda_2 < 0$ for $\mu_1 < 0$ and for all μ_2.

Thus the fixed point $P_1 = (\sqrt{-\mu_1}, 0)$ is a saddle point.

The Jacobian matrix J at the fixed point $P_2 = (-\sqrt{-\mu_1}, 0)$is

$$J = \begin{pmatrix} 0 & 1 \\ -2\sqrt{-\mu_1} & \mu_2 + \sqrt{-\mu_1} \end{pmatrix}$$

The eigenvalues are given by $\lambda = \frac{\mu_2 + \sqrt{-\mu_1} \pm \sqrt{(\mu_2 + \sqrt{-\mu_1})^2 - 8\sqrt{-\mu_1}}}{2}$

Let $\lambda_1 = \frac{\mu_2 + \sqrt{-\mu_1} + \sqrt{(\mu_2 + \sqrt{-\mu_1})^2 - 8\sqrt{-\mu_1}}}{2}$ & $\lambda_2 = \frac{\mu_2 + \sqrt{-\mu_1} - \sqrt{(\mu_2 + \sqrt{-\mu_1})^2 - 8\sqrt{-\mu_1}}}{2}$

The fixed point P_2 is stable for $\mu_2 + \sqrt{-\mu_1} < 0$ i.e. $\mu_2 < -\sqrt{-\mu_1}$and unstable for$\mu_2 > \sqrt{-\mu_1}$.

Next, we verify the conditions for Hopf bifurcations at $\mu_2 = -\sqrt{-\mu_1}$.

(1). For $\mu_2 = -\sqrt{-\mu_1}$, $\mu_1 < 0$, the eigenvalues on $(x_-, 0)$ are given by

$$\lambda_{1,2} = \pm i\sqrt{2\sqrt{-\mu_1}}\left(= \pm i\sqrt{-2x_-}\right)$$

(2). Also we have $\left.\frac{d}{db}\left(Re\lambda(\mu_2)\right)\right|_{\mu_2=\mu_{2_0}} = 1 \neq 0$

(3). Stability of the Hopf bifurcation:
To study the stability of the Hopf bifurcation, we change coordinates twice, first to bring the point $(-\sqrt{-\mu_1}, 0)$ to the origin and then to put the vector field into standard form.
Letting $\bar{x} = x - x_{\_}$ and $\bar{y} = y$, we obtain

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2x_{\_} & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{x}^2 - \bar{x}\bar{y} \end{pmatrix} \quad (18)$$

The coefficient matrix of the linear part is $A = \begin{pmatrix} 0 & 1 \\ -2\sqrt{-a} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2x_{\_} & 0 \end{pmatrix}$

The eigenvalues are $\lambda = \pm i\sqrt{-2x_{\_}} (= \pm i\omega)$ where $\omega = \sqrt{-2x_{\_}}$

Therefore the eigenvectors corresponding to the eigenvalue λ are $\begin{pmatrix} 1 \\ \pm i\sqrt{-2x_{\_}} \end{pmatrix}$

Then, using the linear transformation

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix}$$

where $T = \begin{pmatrix} 0 & 1 \\ \sqrt{-2x_{\_}} & 0 \end{pmatrix}$ is the matrix of real and imaginary parts of the eigenvectors of theeigenvalues $\lambda = \pm i\sqrt{-2x_{\_}}$, we obtain the system with linear part in standard form as:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{-2x_{\_}} \\ \sqrt{-2x_{\_}} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$$

where the nonlinear terms are

$$f(u,v) = \frac{1}{\sqrt{-2x_{\_}}} v^2 - uv, \qquad g(u,v) = 0$$

and we have the following:

$f_u = -v$, $f_{uu} = 0$, $f_{uuu} = 0$, $\quad f_{uv} = -1$, $f_{uvv} = 0$, $f_v = \frac{1}{\sqrt{-2x_{\_}}} 2v - u$, $f_{vv} = \frac{2}{\sqrt{-2x_{\_}}}$,

$g_u = 0$, $g_{uu} = 0$, $g_{uuv} = 0$, $\quad g_v = 0$, $g_{vv} = 0$, $g_{vvv} = 0$, $g_{uv} = 0$

Substituting these in the relation

$$a = \frac{1}{16}[f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv}] + \frac{1}{16\omega}[f_{uv}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu} g_{uu} + f_{vv} g_{vv}]$$

We get $a = \frac{1}{16x_{\_}}$ that is $\quad a = -\frac{1}{16\sqrt{-\mu_1}} < 0$ for $\mu_1 < 0 \qquad$ [as f_{xy} denotes $\frac{\partial^2 f}{\partial x \partial y}(0,0)$]

The bifurcation is therefore supercritical and we have a family of stable periodic orbits. Thus, we can conclude that the system undergoes supercritical Hopf bifurcation for parameter values given by $\mu_2 = -\sqrt{-\mu_1}$.
Below, we have given support to our above mentioned theoretical result through numerical simulation. In the first column of Figure 1, figures $a(I)$, $a(II)$ and $a(III)$shows the phase orbitand the vector field and the figures $b(I)$, $b(II)$ and $b(III)$ shows the corresponding time series plot for our considered model for different parameter values and initial points. In *a(I)* and *b(I)*, the parametervalues are taken to be $\mu_1 = -1, \mu_2 = -1.1$ which is just before the occurrence of supercritical Hopf bifurcation at $\mu_1 = -1, \mu_2 = -1$ (according to our theoretical result). Here, we have considered the initial point as$x_0 = 0.4$, $y_0 = -0.01$. The figure *a(I)* shows that the origin is stable and the orbit spirals to it. The same conclusion is supported by the time series plot shown in figure*b(I)*. For the next figures in *a(II)* and *b(II)*, we have considered the parameter values $\mu_1 = -1, \mu_2 = -0.9$ which is just after the occurrence of supercritical Hopf bifurcation. We considered the initial point as $x_0 = -0.66, y_0 = -0.06$, which is inside the limit cycle produced during the Hopf bifurcation.The figure *a(II)* shows that the origin is unstable and the orbit spiral away from it onto the smooth invariant closed curve encircling it (the limit cycle). The same conclusion is supported by the time series plot shown in *b(II)*. In figure *a(III)* and *b(III)* we have kept the parameter values same with the earlier case but changed the initial point as a point outside the limit cycle. Here we have considered $\mu_1 = -1, \mu_2 = -0.9$ with initial point $x_0 = 0.6, y_0 = 0$.Figure *a(III)* and *b(III)* shows that the orbit spiral towards the invariant circle. So, our numerical simulation clearly supports our conclusion that that the system undergoes supercritical Hopf bifurcation for parameter values given by $\mu_2 = -\sqrt{-\mu_1}$.

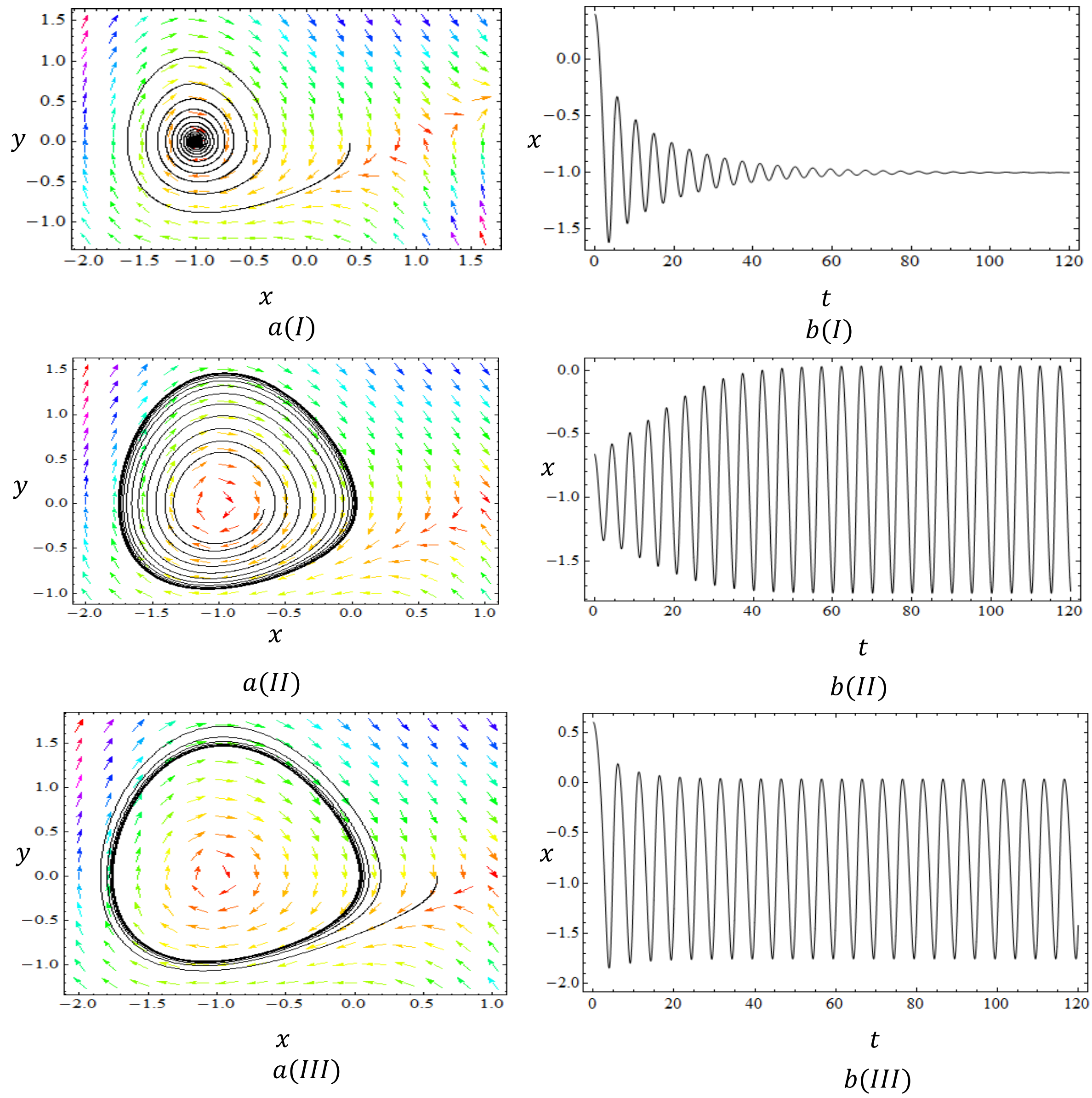

Fig1.The Phase orbit and time series of Eq. (16).a(I) and b(I) before the Hopf bifurcation at the parameter value $\mu_1 = -1, \mu_2 = -1.1$ with initial point $x_0 = 0.4$, $y_0 = -0.01$, the origin is stable and the orbit spirals to it.a(II) and b(II)after the Hopf bifurcation at the parameter value $\mu_1 = -1, \mu_2 = -0.9$ with initial point$x_0 = -0.66, y_0 = -0.06$, which is inside the limit cycle, the origin is unstable and the orbit spiral away from it and onto a smooth invariant closed curve encircling it.a(III) and b(III)after the Hopf bifurcation at the parameter value $\mu_1 = -1, \mu_2 = -0.9$ with initial point $x_0 = 0.6, y_0 = 0$ which is outside the invariant circle(limit cycle), the orbit spiral towards the invariant circle.

Case 2.$\mu_1 = 1, \mu_2 = 1$ (Subcritical Hopf bifurcation)

In this case the system (16) becomes

$$\dot{x} = y$$
$$\dot{y} = \mu_1 + \mu_2 y + x^2 + xy \qquad (19)$$

The fixed points of (19) are given by

$$(x, y) = \left(\pm\sqrt{-\mu_1}, 0\right)$$

which exists for $\mu_1 \leq 0$. We denote both the fixed as $P_1 = (x_+, 0) = (+\sqrt{-\mu_1}, 0)$and $P_2 = (x_-, 0) = (-\sqrt{-\mu_1}, 0)$, where$x_\pm = \pm\sqrt{-\mu_1}$.

The Jacobian matrix J for the linearized system is

$$J = \begin{pmatrix} 0 & 1 \\ 2x + y & \sqrt{\mu_2} + x \end{pmatrix}$$

The Jacobian matrix J at the fixed point $P_1 = (\sqrt{-\mu_1}, 0)$ is $J = \begin{pmatrix} 0 & 1 \\ 2\sqrt{-\mu_1} & \mu_2 + \sqrt{-\mu_1} \end{pmatrix}$

$$\lambda = \frac{\mu_2 + \sqrt{-\mu_1} \pm \sqrt{(\mu_2 + \sqrt{-\mu_1})^2 + 8\sqrt{-\mu_1}}}{2}$$

Let $\lambda_1 = \frac{\mu_2 + \sqrt{-\mu_1} + \sqrt{(\mu_2 + \sqrt{-\mu_1})^2 + 8\sqrt{-\mu_1}}}{2}$ & $\lambda_2 = \frac{\mu_2 + \sqrt{-\mu_1} - \sqrt{(\mu_2 + \sqrt{-\mu_1})^2 + 8\sqrt{-\mu_1}}}{2}$

We observe that $\lambda_1 > 0$ and $\lambda_2 < 0$ for $\mu_1 < 0$ and for all μ_2.
Thus the fixed point $P_1 = (\sqrt{-\mu_1}, 0)$ is a saddle point.

The Jacobian matrix J at the fixed point $P_2 = (-\sqrt{-\mu_1}, 0)$ is $J = \begin{pmatrix} 0 & 1 \\ -2\sqrt{-\mu_1} & \mu_2 - \sqrt{-\mu_1} \end{pmatrix}$

The eigenvalues are given by $\lambda = \frac{\mu_2 - \sqrt{-\mu_1} \pm \sqrt{(\mu_2 - \sqrt{-\mu_1})^2 - 8\sqrt{-\mu_1}}}{2}$

Let $\lambda_1 = \frac{\mu_2 - \sqrt{-\mu_1} + \sqrt{(\mu_2 - \sqrt{-\mu_1})^2 - 8\sqrt{-\mu_1}}}{2}$ & $\lambda_2 = \frac{\mu_2 - \sqrt{-\mu_1} - \sqrt{(\mu_2 - \sqrt{-\mu_1})^2 - 8\sqrt{-\mu_1}}}{2}$

The fixed point P_2 is stable for $\mu_2 - \sqrt{-\mu_1} < 0$i.e. $\mu_2 < \sqrt{-\mu_1}$ and unstable for $\mu_2 > \sqrt{-\mu_1}$.
Next, we verify the conditions for Hopf bifurcations at $\mu_2 = \sqrt{-\mu_1}$

(1). For $\mu_2 = \sqrt{-\mu_1}$, $\mu_1 < 0$, the eigenvalues on $(x_-, 0)$ are given by

$$\lambda_{1,2} = \pm i\sqrt{2\sqrt{-\mu_1}} (= \pm i\sqrt{-2x_-})$$

(2). Also we have observed that $\frac{d}{db}\left(Re\ \lambda(\mu_2)\right)\Big|_{\mu_2 = \mu_{2_0}} = 1 \neq 0$

(3). Stability of the Hopf bifurcation:
To study the stability of the Hopf bifurcation, we change coordinates twice, first to bring the point $(-\sqrt{-\mu_1}, 0)$ to the origin and then to put the vector field into standard form.
Letting $\bar{x} = x - (-\sqrt{-\mu_1})$and $\bar{y} = y$, we obtain

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\sqrt{-\mu_1} & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{x}^2 + \bar{x}\bar{y} \end{pmatrix}$$

The coefficient matrix of the linear part is $A = \begin{pmatrix} 0 & 1 \\ -2\sqrt{-\mu_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2x_- & 0 \end{pmatrix}$

The eigenvalues are $\lambda = \pm i\sqrt{2\sqrt{-\mu_1}} = \pm i\sqrt{-2x_-} (= \pm i\omega)$ where $\omega = \sqrt{-2x_-}$

Therefore the eigenvectors corresponding to the eigenvalue λ are $\begin{pmatrix} 1 \\ \pm i\sqrt{-2x_-} \end{pmatrix}$

Then, using the linear transformation

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix}$$

where $T = \begin{pmatrix} 0 & 1 \\ \sqrt{-2x_-} & 0 \end{pmatrix}$ is the matrix of real and imaginary parts of the eigenvectors of the eigenvalues $= \pm i\sqrt{-2x_-}$, we obtain the system with linear part in standard form:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{-2x_-} \\ \sqrt{-2x_-} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$$

where the nonlinear terms are

$$f(u,v) = uv + \frac{1}{\sqrt{-2x_-}} v^2$$
$$g(u,v) = 0$$

and we have the following:

$f_u = v$, $f_{uu} = 0$,$f_{uuu} = 0$, $f_{uv} = 1$, $f_{uvv} = 0$, $f_v = u + \frac{1}{\sqrt{-2x_-}} 2v = 0$, $f_{vv} = \frac{2}{\sqrt{-2x_-}}$,
$g_u = 0$, $g_{uu} = 0$, $g_{uuv} = 0$, $g_v = 0$, $g_{vv} = 0$, $g_{vvv} = 0$, $g_{u9v} = 0$

Substituting these in the relation

$$a = \frac{1}{16}[f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv}] + \frac{1}{16\omega}[f_{uv}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}]$$

We get $a = -\frac{1}{16x_-}$ that is $a = \frac{1}{16\sqrt{-\mu_1}} > 0$ for $\mu_1 < 0$ [as f_{xy} denotes $\frac{\partial^2 f}{\partial x\, \partial y}(0,0)$]

The bifurcation is therefore subcritical and we have a family of unstable periodic orbits. Thus, we can conclude that the system undergoes subcritical Hopf bifurcation for parameter values given by $\mu_2 = \sqrt{-\mu_1}$.

Below, we have given support to our above mentioned theoretical result through numerical simulation. In the first column of Figure 2, figures *a(I)*, *a(II)* and *a(III)* shows the phase orbit and the vector field and the figures *b(I)*, *b(II)* and *b(III)* shows the corresponding time series plot for our considered model for different parameter values and initial points. In *a(I)* and *b(I)*, the parameter values are taken to be $\mu_1 = -1, \mu_2 = 0.9$ which is just before the occurrence of subcritical Hopf bifurcation at $\mu_1 = -1, \mu_2 = 1$ (according to our theoretical result) with initial point $x_0 = -1.7,\ y_0 = 0$ which is inside the limit cycle. The figure *a(I)* shows that the origin is stable and the orbit spirals to it. The same conclusion is supported by the time series plot shown in figure *b(I)*. For the next figures in *a(II)* and *b(II)*, we have considered the parameter values $\mu_1 = -1, \mu_2 = -0.9$ which is before the occurrence of subcritical Hopf bifurcation. We considered the initial point as $x_0 = -1.8, y_0 = 0$, which is outside the limit cycle produced during the Hopf bifurcation. The figure *a(II)* shows that the origin is stable and the limit cycle is unstable. The orbit spiral away from the limit cycle. The same conclusion is supported by the time series plot shown in *b(II)*. In figure *a(III)* and *b(III)* we have considered parameter values $\mu_1 = -1, \mu_2 = 1.1$ which is just after the subcritical Hopf bifurcation. The initial point was taken as $x_0 = -0.9, y_0 = 0$. Figure *a(III)* and *b(III)* shows that the origin is unstable, the orbits spiral away from it and no limit cycle exists in the neighbourhood. So, our numerical simulation clearly supports our conclusion that that the system undergoes subcritical Hopf bifurcation for parameter values given by $\mu_2 = \sqrt{-\mu_1}$.

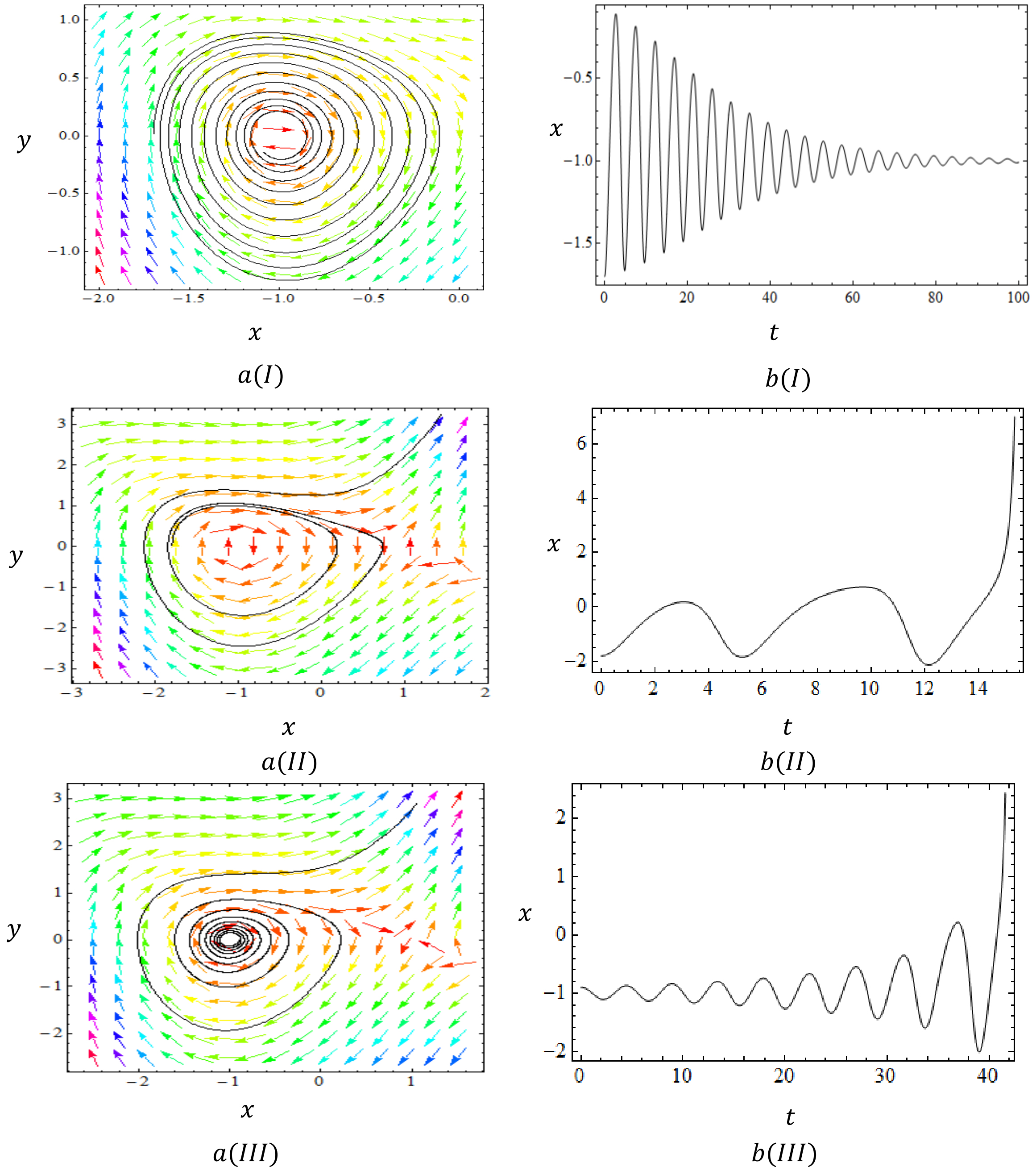

Fig2. The Phase orbit and time series of Eq. (4). a(I) and b(I) before the Hopf bifurcation at the parameter value $\mu_1 = -1, \mu_2 = 0.9$ with initial point $x_0 = -1.7,\ y_0 = 0$,which is inside the limit cycle, the origin is stable and the limit cycle is unstable, the orbit spirals to the origin. a(II) and b(II) before the Hopf bifurcation

at the parameter value $\mu_1 = -1, \mu_2 = -0.9$ with initial point $x_0 = -1.8, y_0 = 0$, which is outside the limit cycle, the origin is stable and limit cycle is unstable and the orbit spiral away from it. (a(III) and b(III)) after the Hopf bifurcation at the parameter value $\mu_1 = -1, \mu_2 = 1.1$ with initial point $x_0 = -0.9, y_0 = 0$, the origin is unstable, the orbits spiral away from it and there is no limit cycle exists.

VI CONCLUSION

We have presented Hopf bifurcation of a nonlinear system, which admits both supercritical and subcritical cases depending upon different parameter values. The results were derived theoretically with the help of center manifold theory and normal form method. Numerical simulations were presented in support of the conclusions drawn theoretically.

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