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Some Common Fixed Point Theorems for Multivalued Mappings in Two Metric Spaces

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Abstract: *In this paper we prove some common fixed point theorems for multivalued mappings in two complete metric spaces.*

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I. Introduction

In 1981, Fisher [1] initiated the study of fixed points on two metric spaces. In1991, Popa [2] proved some theorems on two metric spaces. After this many authors([3]-[7],[8],[9]-[11]) proved many fixed point theorems in two metric spaces. Using the Banach contraction mapping, Nadler [12] introduced the concept of multi-valued contraction mapping and he showed that a multi-valued contraction mapping gives a fixed point in the complete metric space. Later, some fixed points theorems for multifunctions on two complete metric spaces have been proved in [13], [14] and [15]**.**The purpose of this paper is to give some common fixed point theorems for multi-valued mappings in two metric spaces.

Definition1.2 A sequence $\{x_n\}$ in a metric space (X, d) is said to be convergent to a point $x \in X$ if given $\in >0$ there exists a positive integer n_0 such that $d(x_n,x) < \in$ for all $n \geq n_0$.

Definition1.3. A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence in X if given $\in >0$ there exists a positive integer n_0 such that $d(x_m,x_n) < \in$ for all $m,n \geq n_0$.

Definition1.4. A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition1.5 Let X be a non-empty set and $f : X \to X$ be a map. An element x in X is called a fixed point of X if $f(x) = x$.

Definition1.6. Let X be a non-empty set and $f , g : X \to X$ be two maps. An element x in X is called a common fixed point of f and g if $f(x) = g(x) = x$.

Definition1.7. Let (X,d_1) and (Y,d_2) be complete metric spaces and B(X) and B(Y) be two families of all non-empty bounded subsets of X and Y respectively. The function $\delta_1(A,B)$ for A, B in B(X) and $\delta_2(C,D)$ for C,D in B (Y) are defined as follow

$$\delta_1(A,B) = \sup\{ d_1(a,b): a \in A, b\in B \}$$

$$\delta_2(C,D) = \sup\{ d_2(c,d): c \in C, d\in D \}$$

and $\delta(A) = \text{diameter}(A)$

If A consists of a single point a we write δ_1(A,B) = δ_1(a,B). If B also consists of a single point b we write δ_1(A,B) = δ_1(a,B) = δ_1(a,b). It follows immediately that δ_1(A,B) = δ_1(B,A) $\geq$ 0, and δ_1(A,B) $\leq$ δ_1(A,C) + δ_1(C,B) for all A,B in B(X).

Definition1.8. If $\{A_n : n = 1,2,3\ldots\}$ is a sequence of sets in B(X), we say that it converges to the closed set A in B(X) if

(i) Each point a in A is the limit of some convergent sequence $\{a_n \in A_n : n = 1,2,3\ldots\}$,

(ii) For arbitrary $\in > 0$, there exists an integer N such that $A_n \subset A\in$ for n > N where $A\in$ is the union of all open spheres with centres in A and radius $\in$.

The set A is then said to be the limit of the sequence $\{A_n\}$.

Definition1.9. Let f be a multivalued mapping of X into B(X). f is continuous at x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{f(x_n)\}$ in B(X) converges to fx in B(X). If f is continuous at each point $x \in X$, then f is continuous mapping of X into B(X).

Definition1.10. Let T be a multifunction of X into B(X). z is a fixed point of T if Tz = {z}.

Lemma 1.11[16]. If {An} and {Bn} are sequences of bounded subsets of a complete metric space (X, d) which converges to the bounded subsets A and B, respectively, then the sequence {δ(An,Bn)} converges to δ(A,B).

II. Main Results

Theorem 2.1: Let (X, d_1) and (Y, d_2) be two complete metric spaces. Let A, B be mappings of X into B(Y) and S, T be mappings of Y into B(X) satisfying the inequalities.

$$\delta_1(SAx, TBx') \leq \lambda \max\{\delta_1(x,x'), \delta_1(x,SAx), \delta_1(x',TBx'), \tfrac{1}{2}[\delta_1(x,TBx')+\delta_1(SAx, x')], \delta_2(Ax, Bx')\} \quad \text{--------- (1)}$$

$$\delta_2(BSy, ATy') \leq \lambda \max\{\delta_2(y,y'), \delta_2(y,BSy), \delta_2(y',ATy'), \tfrac{1}{2}[\delta_2(y,ATy')+\delta_2(BSy,y')], \delta_2(Sy,Ty')\} \quad \text{---------- (2)}$$

for all x, x′ in X and y, y′ in Y where 0< λ < 1. If one of the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = {w} and Sw = Tw = {z}.

Proof: From (1) and (2) we have

$$\delta_1(SAC, TBD) \leq \lambda \max\{\delta_1(C,D), \delta_1(C,SAC), \delta_1(D,TBD), \tfrac{1}{2}[\delta_1(C,TBD)+\delta_1(SAC,D)], \delta_2(AC,BD)\} \quad \text{--------- (3)}$$

$$\delta_2(BSE, ATF) \leq \lambda \max\{\delta_2(E,F), \delta_2(E,BSE), \delta_2(F,ATF), \tfrac{1}{2}[\delta_2(E, ATF)+\delta_2(BSE,F)], \delta_1(SE,TF)\} \quad \text{---------- (4)}$$

$$\forall\, C,D \in B(X) \text{ and } E,F \in B(Y)$$

Let x_0 be an arbitrary point in X. Then $y_1 \in A(x_0)$ since A: X $\rightarrow$ B(Y), $x_1 \in S(y_1)$ since S : Y $\rightarrow$ B(X) , $y_2 \in B(x_1)$ since B: X $\rightarrow$ B(Y) and $x_2 \in T(y_2)$ since T : Y $\rightarrow$ B(X).

Continuing in this way we get for $n \geq 1$, $y_{2n-1} \in A(x_{2n-2})$, $x_{2n-1} \in S(y_{2n-1})$, $y_{2n} \in B(x_{2n-1})$ and $x_{2n} \in T(y_{2n})$. We define the sequences $\{x_n\}$ in B(X) and $\{y_n\}$ in B(Y) by choosing a point

$x_{2n-1} \in (SATB)^{n-1}SAx = X_{2n-1},\ x_{2n} \in (TBSA)^n x = X_{2n},$

$y_{2n-1} \in A(TBSA)^{n-1}x = Y_{2n-1}$ and $y_{2n} \in B(SATB)^{n-1}SAx = Y_{2n}$ $\quad\forall$ n = 1.2,3, …

Now from (3) we have

$\delta_1(X_{2n+1}, X_{2n}) = \delta_1(SAX_{2n}, TBX_{2n-1})$

$\leq \lambda . \max\{ \delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n},SAX_{2n}), \delta_1(X_{2n-1}, TBX_{2n-1}),$
$½[\delta_1(X_{2n},TBX_{2n-1})+ \delta_1(SAX_{2n}, X_{2n-1})], \delta_2(AX_{2n},BX_{2n-1})\}$
$= \lambda .\max \{ \delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n}, X_{2n+1}), \delta_1(X_{2n-1}, X_{2n}),$
$½[\delta_1(X_{2n},X_{2n})+ \delta_1(X_{2n+1}, X_{2n-1})], \delta_2(Y_{2n+1},Y_{2n}) \}$
$= \lambda .\max\{ \delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n}, X_{2n+1}), \delta_1(X_{2n-1}, X_{2n}),$
$½. \delta_1(X_{2n+1}, X_{2n-1}), \delta_2(Y_{2n+1},Y_{2n}) \}$
$\leq \lambda .\max\{ \delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n}, X_{2n+1}), ½. \delta_1(X_{2n+1}, X_{2n})+ \delta_1(X_{2n}, X_{2n-1}),$
$\delta_2(Y_{2n+1},Y_{2n}) \}$

Hence

$\delta_1(X_{2n+1}, X_{2n}) \leq \lambda . \max\{ \delta_1(X_{2n}, X_{2n-1}), \delta_2(Y_{2n+1},Y_{2n}) \}$ --------------- (5)

Now from (4) we have

$\delta_2(Y_{2n+1},Y_{2n}) = \delta_2(BSY_{2n-1}, ATY_{2n})$
$\leq \lambda .\max\{ \delta_2(Y_{2n}, Y_{2n-1}), \delta_2(Y_{2n-1}, BSY_{2n-1}), \delta_2(Y_{2n}, ATY_{2n}),$
$½[\delta_2(Y_{2n-1}, ATY_{2n})+ \delta_2(Y_{2n},BSY_{2n-1})], \delta_1(SY_{2n-1}, TY_{2n})\}$
$\leq \lambda . \max\{ \delta_2(Y_{2n-1}, Y_{2n}), \delta_1(X_{2n-1}, X_{2n})\}$

Similarly we have

$\delta_1(X_{2n}, X_{2n-1}) \leq \lambda .\max \{ \delta_1(X_{2n-2}, X_{2n-1}), \delta_2(Y_{2n-1}, Y_{2n})\}$ ----------------- (6)

$\delta_2(Y_{2n},Y_{2n-1}) \leq \lambda .\max \{ \delta_2(Y_{2n-2}, Y_{2n-1}), \delta_1(X_{2n-1}, X_{2n-2})\}$

from inequalities (5) and (6), we have

$\delta_1(X_{n+1},X_n) \leq \lambda \max \{ \delta_1(X_n, X_{n-1}), \delta_2(Y_{n+1}, Y_n) \}$
$\vdots$
$\leq \lambda^n.\max \{ \delta_1(X_1, x_0), \delta_2(Y_2, Y_1) \} \to 0$ as $n\to\infty$

Also $\delta_1(x_{n+1},x_n) \leq \delta_1(X_{n+1},X_n)$

which implies $\delta_1(x_{n+1},x_n) \to 0$ as $n\to\infty$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, it converges to a point z in X.
Further

$\delta_1(z, X_n) \leq \delta_1(z, x_n)+ \delta_1(x_n, X_n)$
$\leq \delta_1(z, x_n)+ 2\delta_1(X_n, X_{n+1})$

$\Rightarrow \delta_1(z, X_n) \to 0$ as $n \to \infty$

Similarly $\{y_n\}$ is a Cauchy sequence in Y and it converges to a point w in Y and $\delta_2(w, Y_n) \to 0$ as $n \to \infty$.

Now

$\delta_1(SAx_{2n},z) \leq \delta_1(SAx_{2n}, x_{2n}) + \delta_1(x_{2n}, z)$
$\leq \delta_1(SAx_{2n}, TBx_{2n-1}) + \delta_1(x_{2n}, z)$
$\leq \lambda . \max\{ \delta_1(x_{2n}, x_{2n-1}), \delta_1(z,SAx_{2n}), \delta_1(x_{2n}, TBz),$
$½[\delta_1(x_{2n},TBx_{2n-1})+ \delta_1(x_{2n-1},SAx_{2n})], \delta_2(Ax_{2n}, B x_{2n-1})\}$
$+ \delta_1(x_{2n}, z) \to 0$ as $n \to \infty$.

Thus $\lim_{n\to\infty} SAx_{2n} = \{z\} = \lim_{n\to\infty} Sy_{2n+1}$

Similarly we prove

$\lim_{n\to\infty} TBx_{2n-1} = \{z\} = \lim_{n\to\infty} Ty_{2n}$

$\lim_{n\to\infty} \mathbf{B}Sy_{2n-1} = \{w\} = \lim_{n\to\infty} Bx_{2n-1}$

$\lim_{n\to\infty} ATy_{2n} = \{w\} = \lim_{n\to\infty} Ax_{2n}$

Suppose A is continuous, then

$\lim_{n\to\infty} Ax_{2n} = Az = \{w\}$.

we prove SAz = {z}.

We have

$$\delta_1(SAz, z) = \lim_{n\to\infty} \delta_1(SAz, TBx_{2n-1})$$
$$\leq \lim_{n\to\infty} \lambda.\max\{\delta_1(z, x_{2n-1}), \delta_1(z, SAz), \delta_1(x_{2n-1}, TBx_{2n-1}), \tfrac{1}{2}[\delta_1(z, TBx_{2n-1}) + \delta_1(x_{2n-1}, SAz)], \delta_2(Az, Bx_{2n-1})\}$$
$$< \delta_1(z, SAz) \quad (\text{since } \lambda < 1)$$

Thus SAz = {z}.

Hence Sw = {z}. (Since Az = {w})

Now we prove BSw = {w}.

We have

$$\delta_2(BSw, w) = \lim_{n\to\infty} \delta_2(BSw, ATy_{2n})$$
$$\leq \lim_{n\to\infty} \lambda.\max\{\delta_2(w, y_{2n}), \delta_2(w, BSw), \delta_2(y_{2n}, ATy_{2n}), \tfrac{1}{2}[\delta_2(w, ATy_{2n}) + \delta_2(y_{2n}, BSw)], \delta_1(Sw, Ty_{2n})\}$$
$$< \delta_2(w, BSw) \quad (\text{Since } \lambda < 1)$$

Thus BSw = {w}.

Hence Bz = {w}. (Since Sw = z)

Now we prove TBz = {z}.

We have

$$\delta_1(z, TBz) = \lim_{n\to\infty} \delta_1(SAx_{2n}, TBz)$$
$$\leq \lim_{n\to\infty} \lambda.\max\{\delta_1(x_{2n}, z), \delta_1(x_{2n}, SAx_{2n}), \delta_1(z, TBz), \tfrac{1}{2}[\delta_1(x_{2n}, TBz) + \delta_1(z, SAx_{2n})], \delta_2(Ax_{2n}, Bz)\}$$
$$< \delta_1(z, TBz) \quad (\text{Since } \lambda < 1)$$

Thus TBz = { z }.

Hence Tw = {z}. (Since Bz = {w})

Now we prove ATw = {w}.

We have

$$\delta_2(w, ATw) = \lim_{n\to\infty} \delta_2(BSy_{2n-1}, ATw)$$
$$\leq \lim_{n\to\infty} \lambda.\max\{\delta_2(y_{2n-1}, w), \delta_2(y_{2n-1}, BSy_{2n-1}), \delta_2(w, ATw), \tfrac{1}{2}[\delta_2(y_{2n-1}, ATw) + \delta_2(w, BSy_{2n-1})], \delta_1(Sy_{2n-1}, Tw)\}$$
$$< \delta_2(w, ATw) \quad (\text{Since } \lambda < 1)$$

Thus ATw = {w}.

The same results hold if one of the mappings B, S and T is continuous.

Uniqueness: Let z′ be another common fixed point of SA and TB so that z′ is in SAz′ and TBz′.

Using inequalities (1) and (2) we have

$$\max\{\delta_1(SAz', z'), \delta_1(z', TBz')\}$$
$$\leq \delta_1(SAz', TBz')$$
$$\leq \lambda.\max\{\delta_1(z', z'), \delta_1(z', SAz'), \delta_1(z', TBz'), \tfrac{1}{2}[\delta_1(z', TBz') + \delta_1(z', SAz')], \delta_2(Az', Bz')\}$$

$< \lambda . \delta_2(Az',Bz')$

$\leq \lambda . \max\{ \delta_2(ATBz', Bz'), \delta_2(Az', BSAz')\}$

$\leq \lambda . \delta_2(BSAz',ATBz')$

$\leq \lambda^2 . \max\{ \delta_2(Az', Bz'), \delta_2(Az', BSAz'), \delta_2(Bz',AT Bz'),$
$\tfrac{1}{2}[\delta_2(Az',ATBz')+\delta_2(BSAz',Bz')], \delta_1(SAz', TBz') \}$

$< \lambda^2 . \delta_1(SAz', TBz')$

$\Rightarrow$ SAz′ = TBz′ (since $\lambda < 1$)

$\Rightarrow$ SAz′ = TBz′ = {z′} and Az′ = Bz′ = {w′}

Thus SAz′ = TBz′ = Sw′ = Tw′ = {z′} and
BSw′ = ATw′ = Az′ = Bz′ = {w′}

We have

$\delta_1(z, z') = \delta_1(SAz, TBz')$

$\leq \lambda .\max\{ \delta_1(z,z'), \delta_1(z,SAz), \delta_1(z',TBz'),$
$\tfrac{1}{2}[\delta_1(z,TBz')+\delta_1(z',SAz)], \delta_2(Az,Bz')\}$

$< \delta_2(w, w')$

$\delta_2(w, w') = \delta_2(BSw, ATw')$

$\leq \lambda .\max \{ \delta_2(w,w'), \delta_2(w,BSw), \delta_2(w',ATw'),$
$\tfrac{1}{2}[\delta_2(w,ATw')+\delta_2(w',BSw)], \delta_1(Sw,Tw') \}$

$< \delta_1(z, z')$

Hence $\delta_1(z, z') < \delta_2(w, w') < \delta_1(z, z')$

Thus z = z′.

So the point z is the unique common fixed point of SA and TB. Similarly we prove w is a unique common fixed point of BS and AT.

Remark :2.2 : If we put A = B, S = T in the above theorem 2.1, we get the following corollary.

Corollory 2.3 : Let (X, d_1) and (Y, d_2) be two complete metric spaces. Let A be a mapping of X into B(Y) and T be a mapping of Y into B(X) satisfying the inequalities.

$\delta_1(TAx, TAx') \leq \lambda \max \{ \delta_1(x,x'), \delta_1(x,TAx), \delta_1(x',TAx'),$
$\tfrac{1}{2}[\delta_1(x,TAx')+\delta_1(x',TAx)], \delta_2(Ax,Ax') \}$

$\delta_2(ATy, ATy') \leq \lambda \max \{ \delta_2(y,y'), \delta_2(y,ATy), \delta_2(y',ATy'),$
$\tfrac{1}{2}[\delta_2(y,ATy')+\delta_2(y',ATy)], \delta_1(Ty,Ty')\}$

for all x, x′ in X and y, y′ in Y where $0< \lambda < 1$. If one of the mappings A and T is continuous, then TA has a unique fixed point z in X and AT has a unique fixed point w in Y. Further, Az = Bz = {w} and Sw = Tw = {z}.

Theorem 2.4: Let (X, d_1) and (Y, d_2) be two complete metric spaces. Let A, B be mappings of X into B(Y) and S, T be mappings of Y into B(X) satisfying the inequalities.

$\delta_1(SAx, TBx') \leq \lambda \max\{ \delta_1(x,x'), \delta_1(x',TBx'), \delta_2(Ax,Bx'), \tfrac{1}{2}[\delta_1(x,TBx')+\delta_1(SAx,x')],$
$[\delta_1(x,SAx). \delta_1(x',TBx')] / \delta_1(x,x')\}$ --------- (1)

$\delta_2(BSy, ATy') \leq \lambda \max\{ \delta_2(y,y'), \delta_2(y',ATy'), \delta_2(Sy,Ty'), \tfrac{1}{2}[\delta_2(y,ATy')+\delta_2(BSy,y')],$
$[\delta_2(y,BSy). \delta_2(y',ATy')] / \delta_2(y,y')\}$ ------- (2)

for all x, x′ in X and y, y′ in Y where $0< \lambda < 1$. If one of the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = {w} and Sw = Tw = {z}.

Proof:

From (1) and (2) we have

$$\delta_1(SAC, TBD) \leq \lambda \max\{ \delta_1(C,D), \delta_1(D,TBD), \delta_2(AC,BD), \tfrac{1}{2}[\delta_1(C,TBD)+\delta_1(SAC,D)], [\delta_1(C,SAC).\delta_1(D,TBD)] / \delta_1(C,D)\} \quad \text{--------- (3)}$$

$$\delta_2(BSE, ATF)\leq \lambda \max\{ \delta_2(E,F), \delta_2(F,ATF), \delta_1(SE,TF), \tfrac{1}{2}[\delta_2(E,ATF)+\delta_2(BSE,F)], [\delta_2(E,BSE).\delta_2(F,ATF)] / \delta_2(E,F)\} \quad \text{---------- (4)}$$

$$\forall\, C,D \in B(X) \text{ and } E,F \in B(Y)$$

Let x_0 be an arbitrary point in X. Then $y_1 \in A(x_0)$ since $A: X \to B(Y)$, $x_1 \in S(y_1)$ since $S : Y \to B(X)$, $y_2 \in B(x_1)$ since $B: X \to B(Y)$ and $x_2 \in T(y_2)$ since $T : Y \to B(X)$. Continuing in this way we get for $n \geq 1$, $y_{2n-1} \in A(x_{2n-2})$, $x_{2n-1} \in S(y_{2n-1})$, $y_{2n} \in B(x_{2n-1})$ and $x_{2n} \in T(y_{2n})$. We define the sequences $\{x_n\}$ in B(X) and $\{y_n\}$ in B(Y) by choosing a point $x_{2n-1} \in (SATB)^{n-1}SAx = X_{2n-1}$, $x_{2n} \in (TBSA)^n x = X_{2n,}$ $y_{2n-1} \in A\,(TBSA)^{n-1}x = Y_{2n-1}$ and $y_{2n} \in B(SATB)^{n-1}SAx = Y_{2n}$ $\forall$ n = 1.2,3, ...

Now from (3) we have

$$\begin{aligned}
\delta_1(X_{2n+1}, X_{2n}) &= \delta_1(SAX_{2n}, TBX_{2n-1}) \\
&\leq \lambda.\max\{ \delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n-1},TBX_{2n-1}), \delta_2(AX_{2n},BX_{2n-1}), \\
&\qquad \tfrac{1}{2}[\delta_1(X_{2n},TBX_{2n-1})+ \delta_1(SAX_{2n},X_{2n-1})], \\
&\qquad [\delta_1(X_{2n},SAX_{2n}).\ \delta_1(X_{2n-1},TBX_{2n-1})] / \delta_1(X_{2n}, X_{2n-1})\} \\
&= \lambda.\max\{ \delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n-1},X_{2n}), \delta_2(Y_{2n+1},Y_{2n}), \\
&\qquad \tfrac{1}{2}[\delta_1(X_{2n},X_{2n})+ \delta_1(X_{2n+1},X_{2n-1})], \\
&\qquad \delta_1(X_{2n}, X_{2n+1}).\ \delta_1(X_{2n-1}, X_{2n})] / \delta_1(X_{2n}, X_{2n-1})\} \\
&= \lambda.\max\{ \delta_1(X_{2n}, X_{2n-1}), \delta_2(Y_{2n+1},Y_{2n}), 1/2[\delta_1(X_{2n+1},X_{2n})+ \delta_1(X_{2n}, X_{2n-1})], \\
&\qquad \delta_1(X_{2n},X_{2n+1})\} \\
&\leq \lambda.\max\{ \delta_1(X_{2n}, X_{2n-1}), \delta_2(Y_{2n+1},Y_{2n})\} \quad \text{-- (5)}
\end{aligned}$$

Now from (4) we have

$$\begin{aligned}
\delta_2(Y_{2n}, Y_{2n+1}) &= \delta_2(BSY_{2n-1}, ATY_{2n}) \\
&\leq \lambda.\max\{ \delta_2(Y_{2n-1}, Y_{2n}), \delta_2(Y_{2n},ATY_{2n}), \delta_1(SY_{2n-1}, TY_{2n}), \\
&\qquad 1/2[\delta_2(Y_{2n-1},ATY_{2n})+ \delta_2(BSY_{2n-1}, Y_{2n})], \\
&\qquad \delta_2(Y_{2n-1},BSY_{2n-1}).\delta_2(Y_{2n},ATY_{2n}) / \delta_2(Y_{2n-1}, Y_{2n})\} \\
&\leq \lambda.\max\{ \delta_2(Y_{2n-1}, Y_{2n}), \delta_1(X_{2n-1}, X_{2n})\}
\end{aligned}$$

Similarly

$$\delta_1(X_{2n}, X_{2n-1}) \leq \lambda.\max\{ \delta_1(X_{2n-2},X_{2n-1}), \delta_2(Y_{2n-1},Y_{2n})\} \quad \text{---------------------------------- (6)}$$

$$\delta_2(Y_{2n},Y_{2n-1}) \leq \lambda.\max\{ \delta_2(Y_{2n-1}, Y_{2n-2}), \delta_1(X_{2n-1}, X_{2n-2})\}$$

from inequalities (5) and (6), we have

$$\begin{aligned}
\delta_1(X_{n+1},X_n) &\leq \lambda \max\{ \delta_1(X_n, X_{n-1}), \delta_2(Y_{n+1}, Y_n) \} \\
&\ \ \vdots \\
&\leq \lambda^n.\max\{ \delta_1(X_1, x_0), \delta_2(Y_2, Y_1) \} \to 0 \text{ as } n\to\infty
\end{aligned}$$

Also $\delta_1(x_n, x_{n+1}) \leq \delta_1(X_n, X_{n+1})$

$\Rightarrow \delta_1(x_n, x_{n+1}) \to 0$ as $n \to \infty$

Thus $\{x_n\}$ is a Cauchy sequences in X. Since X is complete, $\{x_n\}$ converges to a point z in X. Further

$\delta_1(z, X_n) \le \delta_1(z, x_n) + \delta_1(x_n, X_n)$

$\le \delta_1(z, x_n) + 2\delta_1(X_n, X_{n+1})$

$\Rightarrow \delta_1(z, X_n) \to 0$ as $n \to \infty$

Similarly $\{y_n\}$ is a Cauchy sequence in Y and it converges to a point w in Y and

$\delta_2(w, Y_n) \to 0$ as $n \to \infty$.

Now

$\delta_1(SAx_{2n}, z) \le \delta_1(SAx_{2n}, x_{2n}) + \delta_1(x_{2n}, z)$

$\le \delta_1(SAx_{2n}, TBx_{2n-1}) + \delta_1(x_{2n}, z)$

$\le \lambda.\max\{\delta_1(x_{2n}, x_{2n-1}), \delta_1(x_{2n-1}, TBx_{2n-1}), \delta_2(Ax_{2n}, Bx_{2n-1}),$

$\tfrac{1}{2}[\delta_1(x_{2n}, TBx_{2n-1}) + \delta_1(SAx_{2n}, x_{2n-1})],$

$[\delta_1(x_{2n}, SAx_{2n}).\ \delta_1(x_{2n-1}, TBx_{2n-1})] / \delta_1(x_{2n}, x_{2n-1})\}$

$+ \delta_1(x_{2n}, z) \to 0$ as $n \to \infty$.

Thus $\lim_{n\to\infty} SAx_{2n} = \{z\} = \lim_{n\to\infty} Sy_{2n+1}$

Similarly we prove

$\lim_{n\to\infty} TBx_{2n-1} = \{z\} = \lim_{n\to\infty} Ty_{2n}$

$\lim_{n\to\infty} \mathbf{B}Sy_{2n-1} = \{w\} = \lim_{n\to\infty} Bx_{2n-1}$

$\lim_{n\to\infty} ATy_{2n} = \{w\} = \lim_{n\to\infty} Ax_{2n}$

Suppose A is continuous, then

$\lim_{n\to\infty} Ax_{2n} = Az = \{w\}$.

Now we prove SAz = {z}

We have

$\delta_1(SAz, z) \le \lim_{n\to\infty} \delta_1(SAz, TBx_{2n-1})$

$\le \lim_{n\to\infty} \lambda.\max\{\delta_1(z, x_{2n-1}), \delta_1(x_{2n-1}, TB\,x_{2n-1}), \delta_2(Az, Bx_{2n-1}),$

$\tfrac{1}{2}[\delta_1(z, TBx_{2n-1}) + \delta_1(SAz, x_{2n-1})],$

$\delta_1(z, SAz), \delta_1(x_{2n-1}, TB\,x_{2n-1}) / \delta_1(z, x_{2n-1})\}$

$< \delta_1(z, SAz)$ (Since $\lambda < 1$)

Thus SAz = {z} = Sw

Now we prove BSw = w.

We have

$\delta_2(BSw, w) \le \lim_{n\to\infty} \delta_2(BSw, ATy_{2n})$

$\le \lim_{n\to\infty} \lambda.\max\{\delta_2(w, y_{2n}), \delta_2(y_{2n}, ATy_{2n}), \delta_1(Sw, Ty_{2n}),$

$\tfrac{1}{2}[\delta_2(w, ATy_{2n}) + \delta_2(BSw, y_{2n})],$

$\delta_2(w, BSw).\ \delta_2(y_{2n}, ATy_{2n}) / \delta_2(w, y_{2n})\}$

$< \delta_2(w, BSw)$ (Since $\lambda < 1$)

Thus BSw = {w} = Bz.

Now we prove TBz = {z}.

$\delta_1(z, TBz) \le \lim_{n\to\infty} \delta_1(SAx_{2n}, TBz)$

$\le \lim_{n\to\infty} \lambda.\max\{\delta_1(x_{2n}, z), \delta_1(z, TBz), \delta_2(Ax_{2n}, Bz),$

$\frac{1}{2}[\delta_1(x_{2n}, TBz) + \delta_1(SAx_{2n}, z)],$

$\delta_1(x_{2n}, SAx_{2n}).\ \delta_1(z, TBz) / \delta_1(x_{2n}, z)\}$

$< \delta_1(z, TBz)$ (Since $\lambda < 1$)

Thus TBz = {z}= Tw.

Now we prove ATw = {w}.

$$\delta_2(w, ATw) \le \lim_{n\to\infty} \delta_2(BSy_{2n-1}, ATw)$$

$$\le \lim_{n\to\infty} \lambda.\max\{\delta_2(y_{2n-1}, w), \delta_2(w, ATw), \delta_1(Sy_{2n-1}, Tw),$$

$$\tfrac{1}{2}[\delta_2(y_{2n-1}, ATw) + \delta_2(BSy_{2n-1}, w)],$$

$$\delta_2(y_{2n-1}, BSy_{2n-1}).\ \delta_2(w, ATw) / \delta_2(y_{2n-1}, w)\}$$

$$< \delta_2(w, ATw) \quad \text{(Since } \lambda < 1)$$

Thus ATw = {w}.

The same results hold if one of the mappings B, S and T is continuous.

Uniqueness: Let z′ be another common fixed point of SA and TB so that z′ is in SAz′ and TBz′. Using inequalities (1) and (2) we have

$$\max\{\delta_1(SAz', z'), \delta_1(z', TBz')\}$$

$$\le \delta_1(SAz', TBz')$$

$$\le \lambda.\max\{\delta_1(z', z'), \delta_1(z', TBz'), \delta_2(Az', Bz'),$$

$$\tfrac{1}{2}[\delta_1(z', TBz') + \delta_1(SAz', z')], [\delta_1(z', SAz').\delta_1(z', TBz')] / \delta_1(z', z')\}$$

$$\le \lambda.\ \delta_2(Az', Bz')$$

$$\le \lambda.\max\{\delta_2(ATBz', Bz'), \delta_2(Az', BSAz')\}$$

$$\le \lambda.\ \delta_2(BSAz', ATBz')$$

$$\le \lambda^2.\max\{\delta_2(Az', Bz'), \delta_2(Bz', ATBz'), \delta_1(SAz', TBz')$$

$$\tfrac{1}{2}[\delta_2(z', BSAz') + \delta_2(ATBz', z')],$$

$$\delta_2(z', BSAz').\delta_2(z', ATBz') / \delta_2(z', z')\}$$

$$\le \lambda^2\ \delta_1(SAz', TBz')$$

$\Rightarrow$ SAz′ = TBz′ (since $\lambda < 1$)

$\Rightarrow$ SAz′ = TBz′ = {z′} and Az′ = Bz′ = {w′}

Thus SAz′ = TBz′ = Sw′ = Tw′ = {z′} and

BSw′ = ATw′ = Az′ = Bz′ = {w′}

We have

$$\delta_1(z, z') = \delta_1(SAz, TBz')$$

$$\le \lambda.\max\{\delta_1(z, z'), \delta_1(z', TBz'), \delta_2(Az, Bz'),$$

$$\tfrac{1}{2}[\delta_1(z, TBz') + \delta_1(SAz, z')], \delta_1(z, SAz).\delta_1(z', TBz') / \delta_1(z, z')\}$$

$$< \delta_2(w, w') \quad \text{(Since } \lambda < 1)$$

Now

$$\delta_2(w, w') = \delta_2(BSw, ATw')$$

$$\le \lambda.\max\{\delta_2(w, w'), \delta_2(w', ATw'), \delta_1(Sw, Tw'),$$

$$\tfrac{1}{2}[\delta_2(w, ATw') + \delta_2(BSw, w')], \delta_2(w, BSw).\delta_2(w', ATw') / \delta_2(w, w')\}$$

$$< \delta_1(z, z')$$

Hence $\delta_1(z, z') < \delta_2(w, w') < \delta_1(z, z')$

$\Rightarrow z = z'$.

So the point z is the unique common fixed point of SA and TB. Similarly we prove w is a unique common fixed point of BS and AT.

Remark: 2.5 : If we put A = B, S = T in the above theorem 2.4, we get the following corollary.

Corollory 2.6 : Let (X, d_1) and (Y, d_2) be two complete metric spaces. Let A be a mapping of X into B(Y) and T be a mapping of Y into B(X) satisfying the inequalities.

$$\delta_1(TAx, TAx') \le \lambda \max\{\delta_1(x,x'), \delta_1(x',TAx'), \delta_2(Ax,Ax'), ½[\delta_1(x,TAx')+\delta_1(TAx,x')], [\delta_1(x,TAx).\delta_1(x',TAx')] / \delta_1(x,x')\}$$

$$\delta_2(ATy, ATy') \le \lambda \max\{\delta_2(y,y'), \delta_2(y',ATy'), \delta_2(Ty,Ty'), ½[\delta_2(y,ATy')+\delta_2(ATy,y')], [\delta_2(y,ATy).\delta_2(y',ATy')] / \delta_2(y,y')\}$$

for all x, x′ in X and y, y′ in Y where $0< \lambda < 1$. If one of the mappings A and T is continuous, then TA has a unique fixed point z in X and AT has a unique fixed point w in Y. Further, Az = Bz = {w} and

Sw = Tw = {z}.

Theorem 2.7: Let (X, d_1) and (Y, d_2) be complete metric spaces. Let A, B be mappings of X into B(Y) and S, T be mappings of Y into B(X) satisfying the inequalities.

$$\delta_1(SAx, TBx') \le \lambda . \max\{\delta_1(x,x'), \delta_1(x,SAx), \delta_1(x',TBx'), \delta_2(Ax,Bx'), \frac{\delta_1(x,TBx')}{2}, \frac{\delta_1(SAx,x')}{2}\} \text{ ------- (1)}$$

$$\delta_2(BSy, ATy') \le \lambda . \max\{\delta_2(y,y'), \delta_2(y,BSy), \delta_2(y',ATy'), \delta_1(Sy,Ty'), \frac{\delta_2(y,ATy')}{2}, \frac{\delta_2(BSy,y')}{2}\} \text{ ---------(2)}$$

for all x, x′ in X and y,y′ in Y where $0 \le \lambda < 1$. If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = {w} and Sw = Tw = {z}.

Proof: From (1) and (2) we have

$$\delta_1(SAC, TBD) \le \lambda \max\{\delta_1(C,D), \delta_1(C,SAC), \delta_1(D,TBD), \delta_2(AC,BD), \frac{\delta_1(C,TBD)}{2}, \frac{\delta_1(SAC,D)}{2}\} \text{ --------- (3)}$$

$$\delta_2(BSE, ATF) \le \lambda \max\{\delta_2(E,F), \delta_2(E,BSE), \delta_2(F,ATF), \delta_1(SE,TF), \frac{\delta_2(E,ATF)}{2}, \frac{\delta_2(BSE,F)}{2}\} \text{ ---------- (4)}$$

$\forall$ C,D $\in$ B(X) and E,F $\in$ B(Y)

Let x_0 be an arbitrary point in X. Then $y_1 \in A(x_0)$ since A: X $\to$ B(Y) , $x_1 \in S(y_1)$ since S : Y $\to$ B(X) , $y_2 \in B(x_1)$ since B: X $\to$ B(Y) and $x_2 \in T(y_2)$ since T : Y $\to$ B(X). continuing in this way we get for $n \ge 1$, $y_{2n-1} \in A(x_{2n-2})$, $x_{2n-1} \in S(y_{2n-1})$, $y_{2n} \in B(x_{2n-1})$ and $x_{2n} \in T(y_{2n})$. We define the sequences $\{x_n\}$ in B(X) and $\{y_n\}$ in B(Y) by choosing a point $x_{2n-1} \in (SATB)^{n-1}SAx = X_{2n-1}$, $x_{2n} \in (TBSA)^n x = X_{2n}$, $y_{2n-1} \in A\,(TBSA)^{n-1}x = Y_{2n-1}$ and $y_{2n} \in B(SATB)^{n-1}SAx = Y_{2n}$ $\forall$ n = 1.2,3, ….

Now from (3) we have

$$\delta_1(X_{2n+1}, X_{2n}) \le \delta_1(SAX_{2n}, TBX_{2n-1})$$

$$\le \lambda .\max\{\delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n},SAX_{2n}), \delta_1(X_{2n-1}, TBX_{2n-1}), \delta_2(AX_{2n},BX_{2n-1}), \frac{\delta_1(X_{2n},TBX_{2n-1})}{2}, \frac{\delta_1(SAX_{2n},X_{2n-1})}{2}\}$$

$= \lambda . \max\{ \delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n}, X_{2n+1}), \delta_1(X_{2n-1}, X_{2n}), \delta_2(Y_{2n+1}, Y_{2n}), \frac{\delta_1(X_{2n}, X_{2n})}{2}, \frac{\delta_1(X_{2n+1}, X_{2n-1})}{2} \}$

$\leq \lambda . \max\{ \delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n}, X_{2n+1}), \delta_1(X_{2n-1}, X_{2n}), \delta_2(Y_{2n+1}, Y_{2n}), \frac{\delta_1(X_{2n+1}, X_{2n}) + \delta_1(X_{2n}, X_{2n-1})}{2} \}$

$\leq \lambda . \max\{ \delta_1(X_{2n-1}, X_{2n}), \delta_2(Y_{2n+1}, Y_{2n})\}$ --------------------------- (5)

Now from (4) we have

$\delta_2(Y_{2n}, Y_{2n+1}) \leq \delta_2(BSY_{2n-1}, ATY_{2n})$

$\leq \lambda . \max\{ \delta_2(Y_{2n-1}, Y_{2n}), \delta_2(Y_{2n-1}, BSY_{2n-1}), \delta_2(Y_{2n}, ATY_{2n}), \delta_1(SY_{2n-1}, TY_{2n}), \frac{\delta_2(Y_{2n-1}, ATY_{2n})}{2}, \frac{\delta_2(BSY_{2n-1}, Y_{2n})}{2} \}$

$\leq \lambda . \max\{ \delta_2(Y_{2n-1}, Y_{2n}), \delta_1(X_{2n-1}, X_{2n})\}$

Similarly

$\delta_1(X_{2n}, X_{2n-1}) \leq \lambda . \max\{ \delta_1(X_{2n-2}, X_{2n-1}), \delta_2(Y_{2n-1}, Y_{2n})\}$------------------------------- (6)

$\delta_2(Y_{2n}, Y_{2n-1}) \leq \lambda . \max \{ \delta_2(Y_{2n-1}, Y_{2n-2}), \delta_1(X_{2n-1}, X_{2n-2})\}$

from inequalities (5) and (6), we have

$\delta_1(X_n, X_{n+1}) \leq \lambda \max \{ \delta_1(X_n, X_{n-1}) , \delta_2(Y_{n+1}, Y_n) \}$

$\vdots$

$\leq \lambda^n . \max \{ \delta_1(x_0, X_1) , \delta_2(Y_1, Y_2) \} \to 0$ as $n \to \infty$

Also $\delta_1(x_n, x_{n+1}) \leq \delta_1(X_n, X_{n+1})$

$\Rightarrow \delta_1(x_n, x_{n+1}) \to 0$ as $n \to \infty$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to a point z in X. Further

$\delta_1(z, X_n) \leq \delta_1(z, x_n) + \delta_1(x_n, X_n)$

$\leq \delta_1(z, x_n) + 2\delta_1(X_n, X_{n+1})$

$\Rightarrow \delta_1(z, X_n) \to 0$ as $n \to \infty$

Similarly $\{y_n\}$ is a Cauchy sequence in Y and it converges to a point w in Y and $\delta_2(w, Y_n) \to 0$ as $n \to \infty$.

Now

$\delta_1(SAx_{2n}, z) \leq \delta_1(SAx_{2n}, x_{2n}) + \delta_1(x_{2n}, z)$

$\leq \delta_1(SAx_{2n}, TBx_{2n-1}) + \delta_1(x_{2n}, z)$

$\leq \lambda .\max\{ \delta_1(x_{2n}, x_{2n-1}), \delta_1(x_{2n}, SAx_{2n}), \delta_1(x_{2n-1}, TBx_{2n-1}), \delta_2(Ax_{2n}, Bx_{2n-1}), \frac{\delta_1(x_{2n}, TBx_{2n-1})}{2}, \frac{\delta_1(SAx_{2n}, x_{2n-1})}{2} \}$

$+ \delta_1(x_{2n}, z) \to 0$ as $n \to \infty$

Thus $\lim_{n\to\infty} SAx_{2n} = \{z\} = \lim_{n\to\infty} Sy_{2n+1}$

Similarly we prove

$\lim_{n\to\infty} TBx_{2n-1} = \{z\} = \lim_{n\to\infty} Ty_{2n}$

$\lim_{n\to\infty} BSy_{2n-1} = \{w\} = \lim_{n\to\infty} Bx_{2n-1}$

$\lim_{n\to\infty} ATy_{2n} = \{w\} = \lim_{n\to\infty} Ax_{2n}$

Suppose A is continuous, then

$\lim_{n\to\infty} Ax_{2n} = Az = \{w\}$.

Now we prove SAz = {z}

We have

$\delta_1(SAz,z) = \lim_{n\to\infty} \delta_1(SAz, TBx_{2n-1})$

$$\leq \lim_{n\to\infty} \lambda \,.\, \max\{ \delta_1(z, x_{2n-1}),\ \delta_1(z,SAz),\ \delta_1(x_{2n-1},TB\,x_{2n-1}),\ \delta_2(Az, Bx_{2n-1}), \frac{\delta_1(z,TBx_{2n-1})}{2}, \frac{\delta_1(SAz, x_{2n-1})}{2}\}$$

$< \delta_1(z,SAz)$ (Since $\lambda < 1$)

Thus SAz = {z}.

Hence Sw = {z}. (Since Az = {w})

Now we prove BSw = {w}.

We have

$\delta_2(BSw, w) = \lim_{n\to\infty} \delta_2(BSw, ATy_{2n})$

$$\leq \lim_{n\to\infty} \lambda \,.\, \max\{ \delta_2(w, y_{2n}),\ \delta_2(w, BSw),\ \delta_2(y_{2n}, ATy_{2n}),\ \delta_1(Sw, Ty_{2n}), \frac{\delta_2(w, ATy_{2n})}{2}, \frac{\delta_2(BSw, y_{2n})}{2}\}$$

$< \delta_2(w,BSw)$ (Since $\lambda < 1$)

Thus BSw = {w}.

Hence Bz = {w}. (Since Sw = {z})

Now we prove TBz = {z}

$\delta_1(z,TBz) = \lim_{n\to\infty} \delta_1(SAx_{2n}, TBz)$

$$\leq \lim_{n\to\infty} \lambda \,.\, \max\{ \delta_1(x_{2n},z),\ \delta_1(x_{2n},SAx_{2n}),\ \delta_1(z, TBz),\ \delta_2(Ax_{2n}, Bz), \frac{\delta_1(x_{2n}, TBz)}{2}, \frac{\delta_1(SAx_{2n}, z)}{2}\}$$

$< \delta_1(z, TBz)$ (Since $\lambda < 1$)

Thus TBz = {z}.

Hence Tw = {z}. (Since Bz = {w})

Now we prove ATw = {w}.

$\delta_2(w, ATw) = \lim_{n\to\infty} \delta_2(BSy_{2n-1}, ATw)$

$$\leq \lim_{n\to\infty} \lambda \,.\, \max\{ \delta_2(y_{2n-1}, w),\ \delta_2(y_{2n-1}, BSy_{2n-1}),\ \delta_2(w, ATw),\ \delta_1(Sy_{2n-1}, Tw), \frac{\delta_2(y_{2n-1}, ATw)}{2}, \frac{\delta_2(BSy_{2n-1}, w)}{2}\}$$

$< \delta_2(w, ATw)$ (Since $\lambda < 1$)

Thus ATw = {w}.

The same results hold if one of the mappings B, S and T is continuous.

Uniqueness: Let z′ be another common fixed point of SA and TB so that z′ is in SAz′ and TBz′. Using inequalities (1) and (2) we have

$\max\{ \delta_1(SAz', z'),\ \delta_1(z', TB\,z')\}$

$\leq \delta_1(SAz', TBz')$

$\leq \lambda .\max\{ \delta_1(z', z'), \delta_1(z',SA z'), \delta_1(z', TB z'), \delta_2(A z',B z'), \frac{\delta_1(z',TBz')}{2}, \frac{\delta_1(SAz',z')}{2} \}$

$\leq \lambda . \delta_2(Az',Bz')$

$\leq \lambda . \max\{ \delta_2(ATBz', Bz'), \delta_2(Az',BSAz')\}$

$\leq \lambda . \delta_2(BSAz',ATBz')$

$\leq \lambda^2. \max\{ \delta_2(Az', Bz'), \delta_2(Az', BSAz'), \delta_2(Bz', ATBz'), \delta_1(SAz', TBz'), \frac{\delta_2(Az', ATBz')}{2}, \frac{\delta_2(BSAz', Bz')}{2} \}$

$\leq \lambda^2 \delta_1(SAz', TBz')$

$\Rightarrow$ SAz′ = TBz′ (since $\lambda < 1$)

$\Rightarrow$ SAz′ = TBz′ = {z′} and Az′ = Bz′ = {w′}

Thus SAz′ = TBz′ = Sw′ = Tw′ = {z′} and
BSw′ = ATw′ = Az′ = Bz′ = {w′}

We have

$\delta_1(z, z') = \delta_1(SAz, TBz')$

$\leq \lambda . \max\{ \delta_1(z, z'), \delta_1(z,SAz), \delta_1(z',TBz'), \delta_2(Az,Bz'), \frac{\delta_1(z, TBz')}{2}, \frac{\delta_1(SAz, z')}{2}\}$

$< \delta_2(w, w')$ (since $\lambda < 1$)

$\delta_2(w, w') = \delta_2(BSw,ATw')$

$\leq \lambda . \max\{ \delta_2(w,w'), \delta_2(w, BSw), \delta_2(w',ATw'), \delta_1(Sw,Tw'), \frac{\delta_2(w, ATw')}{2}, \frac{\delta_2(BSw, w')}{2}\}$

$< \delta_1(z, z')$ (since $\lambda < 1$)

Hence $\delta_1(z, z') < \delta_2(w, w') < \delta_1(z, z')$

Thus z = z′.

So the point z is the unique common fixed point of SA and TB. Similarly we prove w is a unique common fixed point of BS and AT.

Remark 2.8: If we put A = B, S = T in the above theorem 2.7, we get the following corollary.

Corollary 2.9: Let (X,d_1) and (Y,d_2) be two complete metric spaces. Let A be a mapping of X into Y and T be a mapping of Y into X satisfying the inequalities.

$$\delta_1(TAx, TAx') \leq \lambda .\max\{ \delta_1(x,x'), \delta_1(x,TAx), \delta_1(x',TAx'), \delta_2(Ax,Ax'), \frac{\delta_1(x,TAx')}{2}, \frac{\delta_1(TAx,x')}{2}\}$$

$$\delta_2(ATy,ATy') \leq \lambda .\max\{ \delta_2(y,y'), \delta_2(y,ATy), \delta_2(y',ATy'), \delta_1(Ty,Ty'), \frac{\delta_2(y,ATy')}{2}, \frac{\delta_2(ATy,y')}{2}\}$$

for all x, x′ in X and y,y′ in Y where $0 \leq \lambda < 1$. If one of the mappings A and T is continuous, then TA has a unique fixed point z in X and AT has a unique fixed point w in Y. Further,
Az = Bz = {w} and Sw = Tw = {z}.

Theorem 2.10: Let (X, d_1) and (Y, d_2) be complete metric spaces. Let A, B be mappings of X into B(Y) and S, T be mappings of Y into B(X) satisfying the inequalities.

$$\delta_1(SAx, TBx') \le \lambda . \max\{\delta_1(x,x'), \delta_1(x,SAx), \delta_2(Ax,Bx'), \frac{\delta_1(x,TBx')}{2}, \frac{\delta_1(SAx,x')}{2}, [\delta_1(x,SAx).\delta_1(x',TBx')] / \delta_1(x,x')\} \text{ ------- (1)}$$

$$\delta_2(BSy, ATy') \le \lambda . \max\{\delta_2(y,y'), \delta_2(y,BSy), \delta_1(Sy,Ty'), \frac{\delta_2(y,ATy')}{2}, \frac{\delta_2(BSy,y')}{2}, [\delta_2(y,BSy).\delta_2(y',ATy')] / \delta_2(y,y')\} \text{ ---------(2)}$$

for all x, x′ in X and y,y′ in Y where $0 \le \lambda < 1$. If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = {w} and Sw = Tw = {z}.

Proof: From (1) and (2) we have

$$\delta_1(SAC, TBD) \le \lambda \max\{\delta_1(C,D), \delta_1(C,SAC), \delta_2(AC,BD), \frac{\delta_1(C,TBD)}{2}, \frac{\delta_1(SAC,D)}{2}, [\delta_1(C,SAC).\ \delta_1(D,TBD)] / \delta_1(C,D)\} \text{ --------- (3)}$$

$$\delta_2(BSE, ATF) \le \lambda \max\{\delta_2(E,F), \delta_2(E,BSE), \delta_1(SE,TF), \frac{\delta_2(E,ATF)}{2}, \frac{\delta_2(BSE,F)}{2}, [\delta_2(E,BSE).\ \delta_2(F',ATF)] / \delta_2(E,F)\} \text{ ---------- (4)}$$

$$\forall\ C,D \in B(X) \text{ and } E,F \in B(Y)$$

Let x_0 be an arbitrary point in X. Then $y_1 \in A(x_0)$ since A: X $\rightarrow$ B(Y), $x_1 \in S(y_1)$ since S : Y $\rightarrow$ B(X), $y_2 \in B(x_1)$ since B: X $\rightarrow$ B(Y) and $x_2 \in T(y_2)$ since T : Y $\rightarrow$ B(X).Continuing in this way we get for $n \ge 1$, $y_{2n-1} \in A(x_{2n-2})$, $x_{2n-1} \in S(y_{2n-1})$, $y_{2n} \in B(x_{2n-1})$ and $x_{2n} \in T(y_{2n})$. We define the sequences $\{x_n\}$ in B(X) and $\{y_n\}$ in B(Y) by choosing a point $x_{2n-1} \in (SATB)^{n-1}SAx = X_{2n-1}$, $x_{2n} \in (TBSA)^n x = X_{2n}$, $y_{2n-1} \in A\,(TBSA)^{n-1}x = Y_{2n-1}$ and $y_{2n} \in B(SATB)^{n-1}SAx = Y_{2n}$ $\forall$ n = 1.2,3, ...

Now from (3) we have

$$\delta_1(X_{2n+1}, X_{2n}) \le \delta_1(SAX_{2n}, TBX_{2n-1})$$

$$\le \lambda .\max\{\delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n},SAX_{2n}), \delta_2(AX_{2n},BX_{2n-1}), \frac{\delta_1(X_{2n},TBX_{2n-1})}{2}, \frac{\delta_1(SAX_{2n},X_{2n-1})}{2}, [\delta_1(X_{2n},SAX_{2n}).\delta_1(X_{2n-1}, TBX_{2n-1})] / \delta_1(X_{2n}, X_{2n-1})\}$$

$$= \lambda . \max\{\delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n}, X_{2n+1}), \delta_2(Y_{2n+1},Y_{2n}), \frac{\delta_1(X_{2n},X_{2n})}{2}, \frac{\delta_1(X_{2n+1},X_{2n-1})}{2}, [\delta_1(X_{2n},X_{2n+1}).\ \delta_1(X_{2n-1},X_{2n})] / \delta_1(X_{2n}, X_{2n-1})\}$$

$$\le \lambda . \max\{\delta_1(X_{2n}, X_{2n-1}), \delta_1(X_{2n}, X_{2n+1}), \delta_2(Y_{2n+1},Y_{2n}), 0, \frac{\delta_1(X_{2n+1},X_{2n}) + \delta_1(X_{2n},X_{2n-1})}{2}, \delta_1(X_{2n},X_{2n+1})\}$$

$$\le \lambda . \max\{\delta_1(X_{2n-1}, X_{2n}), \delta_2(Y_{2n+1}, Y_{2n})\} \text{ --------------------------- (5)}$$

Now from (4) we have

$$\delta_2(Y_{2n}, Y_{2n+1}) \le \delta_2(BSY_{2n-1}, ATY_{2n})$$

$$\le \lambda \max\{\delta_2(Y_{2n-1}, Y_{2n}), \delta_2(Y_{2n-1}, BSY_{2n-1}), \delta_1(SY_{2n-1}, TY_{2n}), \frac{\delta_2(Y_{2n-1}, ATY_{2n})}{2},$$

$$\frac{\delta_2(BSY_{2n-1}, Y_{2n})}{2}, [\delta_2(Y_{2n-1}, BSY_{2n-1}).\ \delta_2(Y_{2n}, ATY_{2n}) / \delta_2(Y_{2n-1}, Y_{2n})\}$$

$$\leq \lambda.\max\{\delta_2(Y_{2n-1}, Y_{2n}), \delta_1(X_{2n-1}, X_{2n})\}$$

Similarly

$\delta_1(X_{2n}, X_{2n-1}) \leq \lambda.\max\{\delta_1(X_{2n-2}, X_{2n-1}), \delta_2(Y_{2n-1}, Y_{2n})\}$------------------------------- (6)

$\delta_2(Y_{2n}, Y_{2n-1}) \leq \lambda.\max\{\delta_2(Y_{2n-1}, Y_{2n-2}), \delta_1(X_{2n-1}, X_{2n-2})\}$

from inequalities (5) and (6) we have

$$\delta_1(X_n, X_{n+1}) \leq \lambda \max\{\delta_1(X_n, X_{n-1}), \delta_2(Y_{n+1}, Y_n)\}$$

$$\vdots$$

$$\leq \lambda^n.\max\{\delta_1(x_0, X_1), \delta_2(Y_1, Y_2)\} \to 0 \text{ as } n \to \infty$$

Also $\delta_1(x_n, x_{n+1}) \leq \delta_1(X_n, X_{n+1})$

$\Rightarrow \delta_1(x_n, x_{n+1}) \to 0$ as $n \to \infty$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to a point z in X.

Further

$$\delta_1(z, X_n) \leq \delta_1(z, x_n) + \delta_1(x_n, X_n)$$

$$\leq \delta_1(z, x_n) + 2\delta_1(X_n, X_{n+1})$$

$\Rightarrow \delta_1(z, X_n) \to 0$ as $n \to \infty$

Similarly $\{y_n\}$ is a Cauchy sequence in Y and it converges to a point w in Y and

$\delta_2(w, Y_n) \to 0$ as $n \to \infty$.

Now

$$\delta_1(SAx_{2n}, z) \leq \delta_1(SAx_{2n}, x_{2n}) + \delta_1(x_{2n}, z)$$

$$\leq \delta_1(SAx_{2n}, TBx_{2n-1}) + \delta_1(x_{2n}, z)$$

$$\leq c.\max\{\delta_1(x_{2n}, x_{2n-1}), \delta_1(x_{2n}, SAx_{2n}), \delta_2(Ax_{2n}, Bx_{2n-1}), \frac{\delta_1(x_{2n}, TBx_{2n-1})}{2},$$

$$\frac{\delta_1(SAx_{2n}, x_{2n-1})}{2}, [\delta_1(x_{2n}, SAx_{2n}).\ \delta_1(x_{2n-1}, TBx_{2n-1})] / \delta_1(x_{2n}, x_{2n-1})\}$$

$$+ \delta_1(x_{2n}, z) \to 0 \text{ as } n \to \infty$$

Thus $\lim_{n\to\infty} SAx_{2n} = \{z\} = \lim_{n\to\infty} Sy_{2n+1}$

Similarly we prove

$\lim_{n\to\infty} TBx_{2n-1} = \{z\} = \lim_{n\to\infty} Ty_{2n}$

$\lim_{n\to\infty} \mathbf{B}Sy_{2n-1} = \{w\} = \lim_{n\to\infty} Bx_{2n-1}$

$\lim_{n\to\infty} ATy_{2n} = \{w\} = \lim_{n\to\infty} Ax_{2n}$

Suppose A is continuous, then

$\lim_{n\to\infty} Ax_{2n} = Az = \{w\}$.

Now we prove SAz = {z}

We have

$$\delta_1(SAz, z) = \lim_{n\to\infty} \delta_1(SAz, TBx_{2n-1})$$

$$\leq \lim_{n\to\infty} \lambda.\max\{\delta_1(z, x_{2n-1}), \delta_1(z, SAz), \delta_2(Az, Bx_{2n-1}), \frac{\delta_1(z, TBx_{2n-1})}{2},$$

$$\frac{\delta_1(SAz, x_{2n-1})}{2}, \delta_1(z, SAz), \delta_1(x_{2n-1}, TBx_{2n-1}) / \delta_1(z, x_{2n-1})\}$$

$< \delta_1(z, SAz)$ (Since $\lambda < 1$)

Thus SAz = {z}.
Hence Sw = {z}. (Since Az = {w})
Now we prove BSw = {w}.
We have

$$\delta_2(BSw, w) = \lim_{n\to\infty} \delta_2(BSw, ATy_{2n})$$

$$\leq \lim_{n\to\infty} \lambda \max\{ \delta_2(w, y_{2n}), \delta_2(w, BSw), \delta_1(Sw, Ty_{2n}), \frac{\delta_2(w, ATy_{2n})}{2}, \frac{\delta_2(BSw, y_{2n})}{2}, \delta_2(w,BSw).\ \delta_2(y_{2n}, ATy_{2n}) / \delta_2(w, y_{2n})\}$$

$< \delta_2(w,BSw)$ (Since $\lambda < 1$)

Thus BSw = {w}.
Hence Bz = {w}. (Since Sw = {z})
Now we prove TBz = {z}

$$\delta_1(z,TBz) = \lim_{n\to\infty} \delta_1(SAx_{2n}, TBz)$$

$$\leq \lim_{n\to\infty} \lambda . \max\{ \delta_1(x_{2n},z), \delta_1(x_{2n},SAx_{2n}), \delta_2(Ax_{2n}, Bz), \frac{\delta_1(x_{2n}, TBz)}{2}, \frac{\delta_1(SAx_{2n}, z)}{2}, \delta_1(x_{2n},SAx_{2n}).\ \delta_1(z,TBz) / \delta_1(x_{2n},z)\}$$

$< \delta_1(z, TBz)$ (Since $\lambda < 1$)

Thus TBz = {z}.
Hence Tw = {z}. (Since Bz = {w})
Now we prove ATw = {w}.

$$\delta_2(w, ATw) = \lim_{n\to\infty} \delta_2(BSy_{2n-1}, ATw)$$

$$\leq \lim_{n\to\infty} \lambda .\max\{ \delta_2(y_{2n-1}, w), \delta_2(y_{2n-1}, BSy_{2n-1}), \delta_1(Sy_{2n-1}, Tw), \frac{\delta_2(y_{2n-1}, ATw)}{2}, \frac{\delta_2(BSy_{2n-1}, w)}{2}, \delta_2(y_{2n-1}, BSy_{2n-1}).\ \delta_2(w,ATw) / \delta_2(y_{2n-1}, w)\}$$

$< \delta_2(w, ATw)$ (Since $\lambda < 1$)

Thus ATw = {w}.
The same results hold if one of the mappings B, S and T is continuous.

Uniqueness: Let z′ be another common fixed point of SA and TB so that z′ is in SAz′ and TBz′.
Using inequalities (1) and (2) we have

$$\max\{ \delta_1(SAz', z'), \delta_1(z', TBz')\}$$

$$\leq \delta_1(SAz', TBz')$$

$$\leq \lambda .\max\{ \delta_1(z', z'), \delta_1(z',SA z'), \delta_2(A z',B z'), \frac{\delta_1(z', TBz')}{2}, \frac{\delta_1(SAz', z')}{2}, [\delta_1(z',SAz').\ \delta_1(z',TBz')] / \delta_1(z', z')\}$$

$$\leq \lambda .\ \delta_2(Az',Bz')$$

$$\leq \lambda .\max\{ \delta_2(ATBz', Bz'), \delta_2(Az',BSAz')\}$$

$$\leq \lambda .\ \delta_2(BSAz',ATBz')$$

$$\leq \lambda^2.\max\{\delta_2(Az', Bz'), \delta_2(Az', BSAz'), \delta_1(SAz', TBz'), \frac{\delta_2(Az', ATBz')}{2},$$

$$\frac{\delta_2(BSAz', Bz')}{2}, \delta_2(z', BSAz').\delta_2(z', ATBz')/\delta_2(z', z')\}$$

$$\leq \lambda^2 \delta_1(SAz', TBz')$$

$\Rightarrow$ SAz′ = TBz′ (since λ < 1)

$\Rightarrow$ SAz′ = TBz′ = {z′} and Az′ = Bz′ = {w′}

Thus SAz′ = TBz′ = Sw′ = Tw′ = {z′} and
BSw′ = ATw′ = Az′ = Bz′ = {w′}

We have

$$\delta_1(z, z') = \delta_1(SAz, TBz')$$

$$\leq \lambda.\max\{\delta_1(z, z'), \delta_1(z, SAz), \delta_2(Az, Bz'), \frac{\delta_1(z, TBz')}{2},$$

$$\frac{\delta_1(SAz, z')}{2}, \delta_1(z, SAz).\delta_1(z', TBz')/\delta_1(z, z')\}$$

$$< \delta_2(w, w') \text{ (since } \lambda < 1)$$

$$\delta_2(w, w') = \delta_2(BSw, ATw')$$

$$\leq \lambda.\max\{\delta_2(w, w'), \delta_2(w, BSw), \delta_1(Sw, Tw'), \frac{\delta_2(w, ATw')}{2},$$

$$\frac{\delta_2(BSw, w')}{2}, \delta_2(w, BSw).\delta_2(w', ATw')/\delta_2(w, w')\}$$

$$< \delta_1(z, z') \text{ (since } \lambda < 1)$$

Hence $\delta_1(z, z') < \delta_2(w, w') < \delta_1(z, z')$

Thus z = z′.

So the point z is the unique common fixed point of SA and TB. Similarly we prove w is a unique common fixed point of BS and AT.

Remark :2.11 : If we put A = B, S = T in the above theorem 2.10, we get the following corollary.

Corollary 2.12: Let (X,d_1) and (Y,d_2) be two complete metric spaces. Let A, B be mappings of X into B(Y) and S, T be mappings of Y into B(X) satisfying the inequalities.

$$\delta_1(TAx, TAx') \leq \lambda.\max\{\delta_1(x,x'), \delta_1(x,TAx), \delta_2(Ax,Ax'), \frac{\delta_1(x,TAx')}{2}, \frac{\delta_1(TAx,x')}{2},$$
$$[\delta_1(x,TAx).\ \delta_1(x',TAx')]/\delta_1(x,x')\}$$

$$\delta_2(ATy, ATy') \leq \lambda.\max\{\delta_2(y,y'), \delta_2(y,ATy), \delta_1(Ty,Ty'), \frac{\delta_2(y,ATy')}{2}, \frac{\delta_2(ATy,y')}{2},$$
$$[\delta_2(y,ATy).\ \delta_2(y',ATy')]/\delta_2(y,y')\}$$

for all x, x′ in X and y, y′ in Y where $0 \leq \lambda < 1$. If one of the mappings A and T is continuous, then TA has a unique fixed point z in X and AT has a unique fixed point w in Y. Further,
Az = Bz = {w} and Sw = Tw = {z}.

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