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Control Chart Analysis of E_k/M/1 Queueing Model

T.Poongodi[1], Dr. (Mrs.) S. Muthulakshmi[2]

[1, 2] *(Assistant Professor, Faculty of Engineering, Professor, Faculty of Science, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, India)*

Abstract: *Queueing problems are most common features not only in our daily-life situations such as bank counters, post offices, ticket booking centres, public transportation systems, but also in more technical environments such as in manufacturing, computer networking and telecommunications. For any queueing system average queue length, average system length, average waiting time in the queue and average waiting time in the system are the main observable performance characteristics. Control chart is a graph used to study how a process changes over time and it is also used to control ongoing processes. In this paper control limits are established to study the behavior of E_k /M /1 queueing model using the performance characteristics. Numerical results are given to highlight its applications.*

Keywords: *queue length, waiting time, Erlang arrival, exponential service, control limits.*

I. Introduction

Queueing model with Erlang arrival has been discussed by Gross and Harris (1998) and several others. Control chart is a quality control technique evolved initially to monitor production processes. Montgomery (2005) proposed a number of applications of Shewhart control charts in assuring quality in manufacturing industries. Shore (2000) developed control chart for random queue length of M /M /s queueing model by considering the first three moments and also Shore (2006) developed Shewhart-like general control charts for G/G/S queueing system using skewness. Khaparde and Dhabe (2010) constructed the control chart for random queue length of M/M/1 queueing model using method of weighted variance. Poongodi and Muthulakshmi (2012) analyzed number of customers in system of M/E_k/1 queueing model using control chart technique. The Erlang arrival queueing system is applicable to many real-life situations:

(i) In military recruits a recruit first lines up to have blood test at one station, an eye examination at the next station, mental test by a psychiatrist at the third, and is examined by a doctor for medical problems at the fourth and so on. Recruit needs to pass all phases before entering for selection.
(ii) In a typical polling system, voters have to pass through many phases like showing identity cards, signing, getting ink mark etc., before voting.
(iii) In an airline counter, passengers are expected to check in which consists of many phases, before entering into the plane.
and so on.

This motivated the author to study the construction of Shewhart control charts for number of customers in the queue, number of customers in the system, waiting time in the queue and waiting time in the system of E_k/M/1 queueing model.

II. E_k/ M/1 Model Description

Consider a queueing system in which the inter-arrival times follow k-Erlang distribution with mean$1/\lambda$ in which an arrival has to pass through k phases, each with a mean time $1/k\lambda$ prior to entering the service. Service times follow an exponential distribution with mean$1/\mu$.

Let p_n be the probability that there are n customers in E_k/M/1 queueing system. Then

$$p_n = \sum_{j=nk}^{nk+k-1} p_j^{(P)} \quad (1)$$

where $p_j^{(P)}$ is the probability of completion of j phases and is given by

$$p_j^{(P)} = \frac{k\lambda\, p_0^{(P)}}{\mu} r_0^{\,j-k} \qquad (j \ge k) \qquad (2)$$

where $r_0 \in (0,1)$ is the root of the characteristic equation

$$\mu z^{k+1} - (k\lambda+\mu) z + k\lambda = 0$$

since $p_0^{(P)} = \frac{1-r_0}{k}$, taking $\rho = \lambda/\mu$, equation (2) becomes

$$p_j^{(P)} = \rho(1-r_0)r_0^{\,j-k}$$

Therefore equation (1), reduces to

$$p_n = \rho\,(1-r_0^k)\,(r_0^k)^{n-1}$$

which is a geometric distribution.

Control chart analysis of $E_k/M/1$ queueing model is carried out in the following sections.

III. Number Of Customers In The Queue

Let L_q denote the number of customers in the queue.

The expected number of customers in the queue, $E(L_q)$ is given by

$$E(L_q) = \sum_{n=1}^{\infty}(n-1)\,p_n$$

$$= \frac{\rho\, r_0^k}{(1-r_0^k)} \quad (3)$$

and

$$E(L_q^2) = \sum_{n=1}^{\infty}(n-1)^2 p_n$$

$$= \frac{\rho\, r_0^k(1+r_0^k)}{(1-r_0^k)^2}$$

Then variance of L_q is

$$Var(L_q) = E(L_q^2) - (E(L_q))^2$$

$$= \frac{\rho\, r_0^k(1+r_0^k-\rho\, r_0^k)}{(1-r_0^k)^2} \quad (4)$$

Upper control limit (UCL), central line (CL) and lower control limit (LCL) of Shewhart control chart, under the assumption that the number of customers in the queue follows normal distribution, are given by

$$\left.\begin{aligned} UCL &= E(L_q) + 3\sqrt{Var(L_q)} \\ CL &= E(L_q) \\ LCL &= E(L_q) - 3\sqrt{Var(L_q)} \end{aligned}\right\} \quad (5)$$

The parameters of the control chart are obtained by using (3) and (4) in (5) as

$$UCL = \frac{\rho\, r_0^k + 3\sqrt{\rho\, r_0^k(1+r_0^k-\rho\, r_0^k)}}{(1-r_0^k)}$$

$$CL = \frac{\rho\, r_0^k}{(1-r_0^k)}$$

$$LCL = \frac{\rho\, r_0^k - 3\sqrt{\rho\, r_0^k(1+r_0^k-\rho\, r_0^k)}}{(1-r_0^k)}$$

IV. Number Of Customers In The System

Let L_s denote the number of customers in the system (both in queue and in service).

The expected number of customers in the system is given by

$$E(L_s) = \sum_{n=1}^{\infty} n p_n$$

$$= \sum_{n=1}^{\infty} n\rho\,(1-r_0^k)\,(r_0^k)^{n-1}$$

$$= \frac{\rho}{(1 - r_0^{\,k})} \qquad (6)$$

and $$E(L_s^2) = \sum_{n=1}^{\infty} n^2 p_n$$

$$= \rho(1 - r_0^{\,k}) \sum_{n=1}^{\infty} n^2 (r_0^{\,k})^{n-1}$$

$$= \frac{\rho(1 + r_0^{\,k})}{(1 - r_0^{\,k})^2}$$

Then variance of L_s is

$$Var(L_s) = E(L_s^2) - (E(L_s))^2$$

$$= \frac{\rho(1 + r_0^{\,k} - \rho)}{(1 - r_0^{\,k})^2} \qquad (7)$$

The parameters of Shewhart control chart, under the assumption that the number of customers in the system follows normal distribution, are given by

$$\left.\begin{aligned} UCL &= E(L_s) + 3\sqrt{Var(L_s)} \\ CL &= E(L_s) \\ LCL &= E(L_s) - 3\sqrt{Var(L_s)} \end{aligned}\right\} \qquad (8)$$

Using (6) and (7) in (8), the parameters of the control chart are obtained as

$$UCL = \frac{\rho + 3\sqrt{\rho(1 + r_0^{\,k} - \rho)}}{(1 - r_0^{\,k})}$$

$$CL = \frac{\rho}{(1 - r_0^{\,k})}$$

$$LCL = \frac{\rho - 3\sqrt{\rho(1 + r_0^{\,k} - \rho)}}{(1 - r_0^{\,k})}$$

V. WAITING TIME IN THE QUEUE

Distribution of waiting time of a customer in the queue for the model under study is given by

$$W_q(t) = 1 - r_0^{\,k}\, e^{-\mu(1-r_0^{\,k})t}, \qquad t \ge 0.$$

Then the pdf of waiting time is

$$w_q(t) = \mu\, r_0^{\,k}\, (1 - r_0^{\,k})\, e^{-\mu(1-r_0^{\,k})t}$$

Let w_q denote the waiting time of customers in the queue.

The mean of w_q is

$$E(w_q) = \mu\, r_0^{\,k} (1 - r_0^{\,k}) \int_0^{\infty} t\, e^{-\mu(1-r_0^{\,k})t} dt$$

$$= \frac{r_0^{\,k}}{\mu(1 - r_0^{\,k})} \qquad (9)$$

and $$E(w_q^2) = \mu\, r_0^{\,k} (1 - r_0^{\,k}) \int_0^{\infty} t^2 e^{-\mu(1-r_0^{\,k})t} dt$$

$$= \frac{2\, r_0^{\,k}}{\mu^2 (1 - r_0^{\,k})^2}$$

Then variance of w_q is

$$\mathrm{Var}(w_q) = E(w_q^2) - (E(w_q))^2$$

$$= \frac{r_0^{\,k} (2 - r_0^{\,k})}{\mu^2 (1 - r_0^{\,k})^2} \qquad (10)$$

The parameters of Shewhart control chart, under the assumption that the waiting time of customers in the queue follows normal distribution, are given by

$$\left.\begin{aligned} UCL &= E(w_q) + 3\sqrt{\mathrm{Var}(w_q)} \\ CL &= E(w_q) \\ LCL &= E(w_q) - 3\sqrt{\mathrm{Var}(w_q)} \end{aligned}\right\} \qquad (11)$$

The parameters of the control chart are obtained by using (9) and (10) in (11) as

$$UCL = \frac{r_0^{\,k} + 3\sqrt{r_0^{\,k}(2 - r_0^{\,k})}}{\mu(1 - r_0^{\,k})}$$

$$CL = \frac{r_0^{\,k}}{\mu(1 - r_0^{\,k})}$$

$$LCL = \frac{r_0^{\,k} - 3\sqrt{r_0^{\,k}(2 - r_0^{\,k})}}{\mu(1 - r_0^{\,k})}$$

VI. WAITING TIME IN THE SYSTEM

Let w_s denote the waiting time of customers in the system.
The expected waiting time of customers in the system, $E(w_s)$ is given by

$$E(w_s) = E(L_s)/\lambda$$

$$= \frac{\rho}{\lambda(1 - r_0^{\,k})} \qquad (12)$$

Also the variance of w_s is given by

$$\mathrm{Var}(w_s) = \mathrm{Var}(L_s)/\lambda^2$$

$$= \frac{\rho(1 + r_0^{\,k} - \rho)}{\lambda^2 (1 - r_0^{\,k})^2} \qquad (13)$$

The parameters of Shewhart control chart, under the assumption that the waiting time of customers in the system follows normal distribution, are given by

$$\begin{aligned} UCL &= E(w_s) + 3\sqrt{\mathrm{Var}(w_s)} \qquad (14) \\ CL &= E(w_s) \\ LCL &= E(w_s) - 3\sqrt{\mathrm{Var}(w_s)} \end{aligned}$$

The control chart parameters are obtained by using (12) and (13) in (14) as

$$UCL = \frac{\rho + 3\sqrt{\rho(1 + r_0^{\,k} - \rho)}}{\lambda(1 - r_0^{\,k})}$$

$$CL = \frac{\rho}{\lambda(1 - r_0^{\,k})}$$

$$LCL = \frac{\rho - 3\sqrt{\rho(1 + r_0{}^k - \rho)}}{\lambda\,(1 - r_0{}^k)}$$

VII. NUMERICAL ANALYSIS

Numerical analysis is carried out to analyze the performance of queueing system with reference to the parameters λ, μ and k. As LCL values are negative for the selected values of the parameters, they are considered as zero and therefore not shown in the table as a separate column.

Table gives the control chart parameters fornumber of customers in the queue, number of customers in the system, waiting time of customers in the queue and waiting time of customers in the system for certain selected values of λ, μ and k.

TABLE - CONTROL CHART PARAMETERS FOR L_Q, L_S, W_Q AND W_S

λ	μ	k	ρ	r_0	L_q		L_s		W_q		W_s	
					CL	UCL	CL	UCL	CL	UCL	CL	UCL
4			0.40	0.7690	0.0756	1.0171	0.4756	2.4411	0.0189	0.2119	0.1189	0.6103
4.5			0.45	0.7991	0.1183	1.3421	0.5683	2.7810	0.0263	0.2576	0.1263	0.6180
5	10	7	0.50	0.8259	0.1776	1.7428	0.6776	3.1873	0.0355	0.3099	0.1355	0.6374
5.5			0.55	0.8500	0.2595	2.2430	0.8095	3.6842	0.0472	0.3712	0.1472	0.6699
6			0.60	0.8720	0.3731	2.8792	0.9731	4.3088	0.0622	0.4452	0.1622	0.7181
	15		0.67	0.8851	0.6174	4.1412	1.2904	5.5491	0.0617	0.3909	0.1290	0.5549
	20		0.50	0.8046	0.1862	1.8021	0.6862	3.2428	0.0186	0.1596	0.0686	0.3243
10	25	6	0.40	0.7425	0.0805	1.0595	0.4805	2.4775	0.0081	0.0880	0.0481	0.2478
	30		0.33	0.6920	0.0411	0.7090	0.3707	2.0803	0.0041	0.0553	0.0371	0.2081
	35		0.29	0.6496	0.0232	0.5109	0.3135	1.8612	0.0023	0.0375	0.0314	0.1862
		5	0.60	0.8343	0.4070	3.0796	1.0070	4.5047	0.1357	0.9445	0.3357	1.5016
		10	0.60	0.9045	0.3471	2.7250	0.9471	4.1586	0.1157	0.8485	0.3157	1.3862
3	5	15	0.60	0.9329	0.3271	2.6053	0.9271	4.0424	0.1090	0.8157	0.3090	1.3474
		20	0.60	0.9483	0.3173	2.5466	0.9173	3.9856	0.1058	0.7996	0.3058	1.3286
		25	0.60	0.9580	0.3120	2.5147	0.9120	3.9548	0.1040	0.7908	0.3040	1.3183

Numerical values in the table reveal the following features:
For all the performance measures of $E_k/M/1$ queueing system
- (i) increase in the arrival rate λ, increases the parameters and UCL for fixed values of μ and k.
- (ii) increase in the service rate μ, decreases the parametersCL and UCL for fixed values of λ and k.
- (iii) increase in the number of phases k, decreases the parametersCL and UCL for fixed values of λ and μ.

VIII. CONCLUSION

In this paper control chart technique is applied to analyze the system characteristics of $E_k/M/1$ queueing system. These characteristics provide the arrivals to decide whether to join the system or not. In addition it gives the information on the waiting time.

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