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Between α-closed Sets and Semi α-closed Sets

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Abstract: *In general topology many strong and weak forms of open and closed sets have been defined and studied. Govindappa Navalagi introduced the concept of semi α-open sets which is a weaker form of α-open sets. Semi*α-open set is defined analogously by replacing the closure operator by the generalized closure operator due to Dunham in the definition of semi α-open sets. In this paper we introduce a new class of sets, namely semi*α-closed sets, as the complement of semi*α-open sets. We find characterizations of semi*α-closed sets. We also define the semi*α-closure of a subset. Further we investigate fundamental properties of the semi*α-closure. We define the semi*α-derived set of a subset and study its properties.*

Keywords: *semi α-open set, semi α-closed set, semi*α-open set, semi*α-interior, semi*α-closed set, semi*α-closure.*

I. Introduction

In 1963 Levine [1] introduced the concepts of semi-open sets and semi-continuity in topological spaces. Levine [2] also defined and studied generalized closed sets as a generalization of closed sets. Dunham [3] introduced the concept of generalized closure using Levine's generalized closed sets and defined a new topology τ^* and studied its properties. Njastad [4] introduced the concept of α-open sets in 1965. In 2000 Navalagi [5] introduced the concept of semi α-open sets by considering α-open set instead of open set in the definition of semi-open set. Hakeem A. Othman [6] introduced and studied various concepts concerning semi α-open sets. Since then, various notions of semi α-open sets have been studied. Pasunkili Pandian [7] defined and studied semi*-preclosed sets and investigated its properties. The authors [8] have recently introduced the concept of semi*α-open sets and investigated its properties in the light of already existing concepts and results in topology. The semi*α-interior of a subset has also been defined and its properties studied.

In this paper, we define a new class of sets, namely semi*α-closed sets, as the complement of semi*α-open sets. We further show that the class of semi*α-closed sets is placed between the class of α-closed sets and the class of semi α-closed sets. We find characterizations of semi*α-closed sets. We investigate fundamental properties of semi*α-closed sets. We also define the semi*α-closure of a subset. We also study some basic properties of semi*α-closure. Further we define the semi*α-derived set of a subset and investigate its properties.

II. Preliminaries

Throughout this paper (X, τ) will always denote a topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of the space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A respectively. Also $\mathcal{F}$ denotes the class of all closed sets in the space (X, τ).

Definition 2.1: A subset A of a space X is

(i) ***generalized closed*** (briefly ***g-closed***) [2] if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in X.

(ii) ***generalized open*** (briefly ***g-open***) [2] if $X\backslash A$ is g-closed in X.

Definition 2.2: If A is a subset of X,

(i) the ***generalized closure*** [3] of A is defined as the intersection of all g-closed sets in X containing A and is denoted by $Cl^*(A)$.

(ii) the ***generalized interior*** of A is defined as the union of all g-open subsets of A and is denoted by $Int^*(A)$.

Definition 2.3: A subset A of a topological space (X, τ) is

(i) ***semi-open*** [1] (resp. ***semi*-open*** [9]) if there is an open set U in X such that $U\subseteq A\subseteq Cl(U)$ (resp. $U\subseteq A\subseteq Cl^*(U)$) or equivalently if $A\subseteq Cl(Int(A))$ (resp. $A\subseteq Cl^*(Int(A))$).

(ii) ***α-open***[4] (resp. ***preopen*** [10]) if $A\subseteq Int(Cl(Int(A)))$ (resp. $A\subseteq Int(Cl(A))$).

(iii) ***semi α-open*** [5] (resp. ***semi*α-open*** [8])if there is an α-open set U in X such that $U\subseteq A\subseteq Cl(U)$ (resp. $U\subseteq A\subseteq Cl^*(U)$ or equivalently if $A\subseteq Cl(Int(Cl(Int(A))))$(resp. $A\subseteq Cl^*(\alpha Int(A)$.)
(iv) ***semi-preopen*** [11]***(resp. semi*-preopen*** [7]) if $A\subseteq Cl(\text{Int}(Cl(A)))$ (resp. $A\subseteq Cl^*(pInt(A))$).
(v) ***semi-closed*** [12](resp. ***semi*-closed***[13], ***preclosed*** [10], ***α-closed*** [4] , ***semi α-closed*** [6] and ***semi*preclosed***[7]) if $Int(Cl(A))\subseteq A$ (resp. $Int^*(Cl(A))\subseteq A$, $Cl(Int(A)\subseteq A$, $Cl(Int(Cl(A)))\subseteq A$, $Int(Cl(Int(Cl(A))))\subseteq A$ and $Int^*(pCl(A))\subseteq A)$.
The class of all semi-open (resp. α-open, semi-closed, preclosed, semi-preclosed, semi*-preclosed, semi*-open and semi*-closed) sets is denoted by SO(X, τ) (resp. αO(X,τ) or τ^{α}, SC(X, τ), PC(X, τ), SPC(X, τ), S*PC(X, τ), S*O(X, τ) and S*C(X, τ)).
Definition 2.4.[13] Let $A\subseteq X$. An element $x\in X$ is called a semi*-limit point of A if every semi*-open set in X containing x intersects $A\setminus\{x\}$.
Definition 2.5.[13] The set of all semi*-limit points of A is called the semi*-derived set of A and it is denoted by $D_{s*}(A)$.
The other forms of derived sets are similarly defined. The derived set, semi-derived set, pre-derived set, α-derived set, semi α-derived set, semi*α-derived set, semi*pre-derived set and semi-pre-derived set of A are respectively denoted by D[A], sD[A], pD[A], αD[A], sαD[A], s*αD[A], s*pD[A], and spD[A].
Definition 2.6: [2] A topological space X is $\mathbf{T_{1/2}}$ if every g-closed set in X is closed.
Definition 2.7: A topological space (X, α) is called an α-structure if αO(X,τ)=τ.
Definition 2.8: [3] If (X, τ) is a topological space, let τ* be the topology on X defined by the Kuratowski closure operator Cl^*. That is, $\tau^*=\{U\subseteq X : Cl^*(X\setminus U)=X\setminus U\}$.
Theorem 2.9: [3] If (X, τ) is a topological space, then (X, τ*) is $T_{1/2}$.
Definition 2.10: [14] A space X is ***locally indiscrete*** if every open set in X is closed.
Definition 2.9: [14] A space X is ***extremally disconnected*** if the closure of every open set in X is open.
Definition 2.11: [15] The topology on the set of integers generated by the set S of all triplets of the form {2n-1, 2n, 2n+1} as sub base is called the ***Khalimsky topology*** and it is denoted by κ. The digital line equipped with the Khalimsky topology is called the ***Khalimsky line***. The topological product of two Khalimsky lines (ℤ, κ) is called the ***Khalimsky plane*** ($\mathbb{Z}^2$, κ^2).
Theorem 2.12: [8] In any topological space,
(i) Every α-open set is semi*α-open.
(ii) Every open set is semi*α-open.
(iii) Every semi*-open set is semi*α-open.
(iv) Every semi*α-open set is semi α-open.
(v) Every semi*α-open set is semi*-preopen.
(vi) Every semi*α-open set is semi-preopen.
(vii) Every semi*α-open set is semi-open.
Theorem 2.13: [8] If A is a subset of a topological space, then the following statements are equivalent:
(i) A is semi*α-open.
(ii) $A\subseteq Cl^*(\alpha Int(A))$.
(iii) $Cl^*(A)=Cl^*(\alpha Int(A))$.
Theorem 2.14: [8] In any topological space, arbitrary union semi*α-open sets is semi*α-open.
Theorem 2.15: [8] If A is semi*α-open in X and B is open in X, then $A\cap B$ is semi*α-open in X.
Theorem 2.16: [8] If A is semi*α-open in X and $B\subseteq X$ is such that $\alpha Int(A)\subseteq B\subseteq Cl^*(A)$. Then B is semi*α-open.
Theorem 2.17:[8]If A is any subset of a topological space X, then $s^*\alpha Int(A)$ is precisely the set of all semi*α-interior points of A.
Remark 2.18:[8] : (i) If (X, τ) is a locally indiscrete space,
τ =αO(X, τ)= S*αO(X, τ) = SαO(X, τ)=S*O(X,τ)=SO(X, τ).
(ii) In an α-structure semi*α-open sets and semi*-preopen sets coincide.
(iii) In an extremally disconnected space, the semi*α-open sets and the semi*-preopen sets coincide.

III. Semi*α-Closed Sets

Definition 3.1. The complement of a semi*α-open set is called ***semi*α-closed.*** The class of all semi*α-closed sets in (X, τ) is denoted by S*αC(X, τ) or simply S*αC(X).
Definition 3.2. A subset A of X is called ***semi*α-regular*** if it is both semi*α-open and semi*α-closed.
Theorem 3.3. For a subset A of a topological space X, the following statements are equivalent:
(i) A is semi*α-closed.

(ii) There is an α-closed set F in X such that $Int^*(F)\subseteq A\subseteq F$.
(iii) $Int^*(\alpha Cl(A))\subseteq A$.
(iv) $Int^*(\alpha Cl(A))=Int^*(A)$.
(v) $Int^*(A\cup Cl(Int(Cl(A))))=Int^*(A)$.

Proof: (i)⟹(ii): Suppose A is semi*α-closed. Then $X\backslash A$ is semi*α-open. Then by Definition 2.3(iii), there is an α-open set U in X such that $U\subseteq X\backslash A\subseteq Cl^*(U)$. Taking the complements we get, $X\backslash U\supseteq A\supseteq X\backslash Cl^*(U)$. Since for any subset U in the space X, we have $Int^*(X\backslash U)=X\backslash Cl^*(U)$. Therefore $F\supseteq A\supseteq Int^*(F)$ where $F=X\backslash U$ is α-closed in X.

(ii)⟹(iii): By assumption, there is an α-closed set F such that $Int^*(F)\subseteq A\subseteq F$. Since $A\subseteq F$ and F is α-closed, we have $\alpha Cl(A)\subseteq F$. Hence $Int^*(\alpha Cl(A))\subseteq Int^*(F)\subseteq A$.

(iii)⟹(iv): By assumption, $Int^*(\alpha Cl(A))\subseteq A$. This implies that $Int^*(\alpha Cl(A))\subseteq Int^*(A)$. Since it is true that $A\subseteq \alpha Cl(A)$, we have $Int^*(A)\subseteq Int^*(\alpha Cl(A))$. Therefore $Int^*(\alpha Cl(A))=Int^*(A)$.

(iv)⟹(i): If $Int^*(\alpha Cl(A))=Int^*(A)$, then taking the complements, we get $X\backslash Int^*(\alpha Cl(A))=X\backslash Int^*(A)$. Hence $Cl^*(\alpha Int(X\backslash A))=Cl^*(X\backslash A)$. Therefore by Theorem 2.13., $X\backslash A$ is semi*α-open and hence A is semi*α-closed.

(iv)⟺(v): Follows from the fact that for any subset A, $\alpha Cl(A)=A\cup Cl(Int(Cl(A)))$

Theorem 3.4. If a subset A of a topological space X is semi*α-closed, then the following statements hold:
(i) There is a closed set H in X such that $Int^*(Cl(Int(H)))\subseteq A\subseteq H$.
(ii) There is a closed set H in X such that $Int^*(sInt(H))\subseteq A\subseteq H$.
(iii) There is a closed set H in X such that $Int^*(H\cap Cl(Int(H)))\subseteq A\subseteq H$.
(iv) $Int^*(Cl(Int(Cl(A))))\subseteq A$.

Proof:(i) Since A is semi*α-closed, from Theorem 3.3 there is an α- closed set F such that $Int^*(F)\subseteq A\subseteq F$. Since F is α-closed, there is a closed set H such that $Cl(Int(H))\subseteq A\subseteq H$. This proves (i)
(ii) follows from (i) and from the fact that for any closed set H, $sInt(H)=Cl(Int(H))$
(iii) follows from (ii) since for any set H, $sInt(H)=H\cap Cl(Int(H))$
(iv) Follows from (i) since H is closed.

Remark 3.5. (i) In any topological space (X, τ), ϕ and X are semi*α-closed sets.
(ii) In a $T_{1/2}$ space, the semi*α-closed sets and the semi-α-closed sets coincide. In particular, in the Khalimsky line and in the real line with usual topology the semi*α-closed sets and the semi-α-closed sets coincide.

Theorem 3.6. Arbitrary intersection of semi*α-closed sets is also semi*α-closed.

Proof: Let $\{A_i\}$ be a collection of semi*α-closed sets in X. Since each A_i is semi*α-closed, $X\backslash A_i$ is semi*α-open. By Theorem 2.14, $X\backslash(\cap A_i)=\cup(X\backslash A_i)$ is semi*α-open. Hence $\cap A_i$ is semi*α-closed.

Corollary 3.7. If A is semi*α-closed and U is semi*α-open in X, then $A\backslash U$ is semi*α-closed.

Proof: Since U is semi*α-open, $X\backslash U$ is semi*α-closed. Hence by Theorem 3.6, $A\backslash U=A\cap(X\backslash U)$ is semi*α-closed.

Remark 3.8. Union of two semi*α-closed sets need not be semi*α-closed as seen from the following examples.

Example 3.9: In the space (X, τ), where $X = \{a, b, c, d\}$ and $\tau =\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$, the subsets $\{a, c, e\}$ and $\{b, c, e\}$ are semi*α-closed but their union $\{a, b, c, e\}$ is not semi*α-closed.

Example 3.10: Let X= {1, 2}×{1, 2, 3} be given the subspace topology of the digital plane. In X, the subsets A={(1, 1), (1, 2), (2, 1), (2, 3)} and B={(1, 2), (1, 3), (2, 1), (2, 3)} are semi*α-closed but $A\cup B$ is not semi*α-closed.

Theorem 3.11. If A is semi*α-closed in X and B is closed in X, then $A\cup B$ is semi*α-closed.

Proof: Since A is semi*α-closed, $X\backslash A$ is semi*α-open in X. Also $X\backslash B$ is open. By Theorem 2.15, $(X\backslash A)\cap(X\backslash B)=X\backslash(A\cup B)$ is semi*α-open in X. Hence $A\cup B$ is semi*α-closed in X.

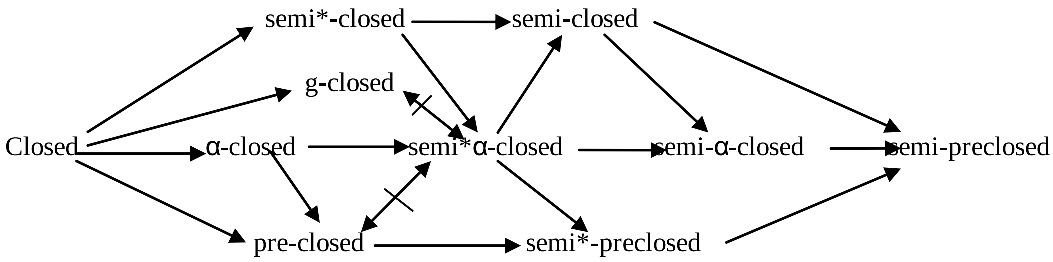
Theorem 3.12. In any topological space,
(i) Every α-closed set is semi*α-closed.
(ii) Every closed set is semi*α-closed.
(iii) Every semi*-closed set is semi*α-closed.
(iv) Every semi*α-closed set is semi-α-closed.
(v) Every semi*α-closed set is semi*-preclosed.
(vi) Every semi*α-closed set is semi-preclosed.
(vii) Every semi*α-closed set is semi-closed.

Proof: (i) Let A be α-closed. Then $X\backslash A$ is α-open. By Theorem 2.12(i), $X\backslash A$ is semi*α-open. Hence A is semi*α-closed. (ii) follows from (i) and the fact that every closed set is α-closed. To prove (iii) let A be a semi*-closed set in X. Then $X\backslash A$ is semi*-open in X. By Theorem 2.12(iii) $X\backslash A$ is semi*α-open. Hence A is semi*α-closed in X. To prove (iv) let A be semi*α-closed, then $X\backslash A$ is semi*α-open. By Theorem 2.12(iv), $X\backslash A$ is semi-α-open.

Hence *A* is semi-α-closed. Let *A* be a semi*α-closed set in *X*. Then $X\setminus A$ is semi*α-open in *X*. By Theorem 2.12(v) $X\setminus A$ is semi*-preopen. Hence *A* is semi*-preclosed in *X*. This proves (v). The statement (vi) follows from (v) and the fact that every semi*-preclosed set is semi-preclosed. (vii) follows from Theorem 2.12(vii).

Remark 3.13: The converse of each of the statements in Theorem 3.12 is not true.

Remark 3.14. From the discussions above we have the following diagram



Relationship Diagram for Semi*α-Closed Sets

Corollary 3.15. (i)If *A* is semi*α-closed and *F* is α-closed in *X*, then $A\cap F$ is semi*α-closed in *X*.
(ii) If *A* is semi*α-closed and *F* is closed in *X*, then $A\cap F$ is semi*α-closed in *X*.
(iii)If *A* is semi*α-closed and *U* is open in *X*, then $A\setminus U$ is semi*α-closed in *X*.

Proof: (i) Since *F* is α-closed, by Theorem 3.12(i) *F* is semi*α-closed. Then by Theorem 3.6, $A\cap F$ is semi*α-closed.
(ii)Since *F* is closed, by Theorem 3.11(ii), *F* is semi*α-closed. Then by Theorem 3.6, $A\cap F$ is semi*α-closed.
(iii)Since *U* is open in *X*, its complement $X\setminus U$ is closed in *X*. From (i), $A\setminus U=A\cap(X\setminus U)$ is semi*α-closed .

Theorem 3.16. In any topological space (X, τ),
(i) $\mathcal{F}\subseteq \alpha C(X, \tau)\subseteq S^*\alpha C(X, \tau)\subseteq S\alpha C(X, \tau)\subseteq SPC(X, \tau)$
(ii) $\mathcal{F}\subseteq \alpha C(X, \tau)\subseteq S^*\alpha C(X, \tau) \subseteq S^*PC(X, \tau) \subseteq SPC(X, \tau)$
(iii) $\mathcal{F}\subseteq \alpha C(X, \tau)\subseteq S^*\alpha C(X, \tau) \subseteq SC(X, \tau) \subseteq SPC(X, \tau)$ and
(iv) $\mathcal{F} \subseteq S^*C(X, \tau) \subseteq S^*\alpha C(X, \tau) \subseteq S\alpha C(X, \tau)\subseteq SPC(X, \tau)$.

Proof: Follows from Theorem 3.11 and from the facts that every closed set is α-closed, every semi-closed set is semi-preclosed, every closed set is semi*-closed, every semi α-closed set is semi-preclosed and every semi*-preclosed set is semi-preclosed.

Remark 3.17. (i) If (X, τ) is a locally indiscrete space, then $\mathcal{F}=\alpha C(X, \tau)= S^*\alpha C(X, \tau) = S\alpha C(X, \tau)=S^*C(X,\tau)= SC(X, \tau)$.
(ii) In the Sierpinski space (X, τ), where $X=\{a, b\}$ and $\tau=\{\phi, \{a\}, X\}$, $\mathcal{F}=\alpha C(X, \tau)=S^*C(X, \tau)=SC(X, \tau)= S^*\alpha C(X, \tau)=S\alpha C(X, \tau)=PC(X, \tau)=S^*PC(X, \tau)=SPC(X, \tau)$.
(iii) If (X, τ) is extremally disconnected, $S^*\alpha C(X, \tau)=S^*PC(X, \tau)$.
(iv) If (X, τ) is an α-structure, $S^*\alpha C(X, \tau) = S^*C(X,\tau)$
(v) The inclusions in Theorem 3.16 involving $S^*\alpha(X,\tau)$ may be strict and equality may also hold. This can be seen from the following examples.

Remark 3.18. The concept of semi*α-closed set is independent of each of the concepts of g-closed sets and pre-closed sets.

Theorem 3.19. If *A* is semi*α-closed in *X* and *B* be a subset of *X* such that $Int^*(A)\subseteq B\subseteq \alpha Cl(A)$, then *B* is semi*α-closed in *X*.

Proof: Since *A* is semi*α-closed, $X\setminus A$ is semi*α-open. Now $Int^*(A)\subseteq B\subseteq \alpha Cl(A)$ which implies $X\setminus Int^*(A)\supseteq X\setminus B\supseteq$
$X\setminus \alpha Cl(A)$. That is, $Cl^*(X\setminus A)\supseteq X\setminus B\supseteq \alpha Int(X\setminus A)$. Therefore by Theorem 2.16, $X\setminus B$ is semi*α-open. Hence *B* is semi*α-closed.

Theorem 3.20. Let **C** be a collection of subsets in *X* satisfying (i) $\alpha C(X) \subseteq \mathbf{C}$ (ii) If $A\in\mathbf{C}$ and $D\subseteq X$ is such that $Int^*(A)\subseteq D\subseteq \alpha Cl(A)$ imply that $D\in\mathbf{C}$. Then $S^*\alpha C(X)\subseteq \mathbf{C}$. Thus $S^*\alpha C(X)$ is the smallest collection of subsets of *X* satisfying the conditions (i) and (ii).

Proof: By Theorem 3.16(i), $S^*\alpha C(X)$ satisfies (i) and by Theorem 3.19, $S^*\alpha C(X)$ satisfies (ii). Further if $A\in S^*\alpha C(X)$, then by Theorem 3.3, there is an α-closed set *F* in *X* such that $Int^*(F)\subseteq A\subseteq F$. By (i), $F\in\mathbf{C}$. Since *F* is α-closed, $\alpha Cl(F)=F$. Therefore $Int^*(F)\subseteq A\subseteq \alpha Cl(F)$. Hence by (ii), $A\in\mathbf{C}$. Thus $S^*\alpha C(X)\subseteq \mathbf{C}$. This shows that $S^*\alpha C(X)$ is the smallest collection of subsets of *X* satisfying (i) and (ii).

IV. Semi*α-Closure Of A Set

Definition 4.1. If A is a subset of a topological space X, the ***semi*α-closure*** of A is defined as the intersection of all semi*α-closed sets in X containing A. It is denoted by $s^*\alpha Cl(A)$.

Theorem 4.2. If A is any subset of a topological space (X, τ), then $s^*\alpha Cl(A)$ is semi*α-closed. In fact $s^*\alpha Cl(A)$ is the smallest semi*α-closed set in X containing A.

Proof: Since $s^*\alpha Cl(A)$ is the intersection of all semi*α-closed supersets of A, by Theorem 3.6, it is semi*α-closed and is contained in every semi*α-closed set containing A and hence it is the smallest semi*α-closed set in X containing A.

Theorem 4.3. Let A be a subset of a topological space (X, τ). Then A is semi*α-closed if and only if $s^*\alpha Cl(A)=A$.

Proof: If A is semi*α-closed, then $s^*\alpha Cl(A)=A$ is obvious. Conversely, let $s^*\alpha Cl(A)=A$. By Theorem 4.2, $s^*\alpha Cl(A)$ is semi*α-closed and hence A is semi*α-closed.

Theorem 4.4. (Properties of Semi*α-Closure)

In any topological space (X, τ), the following results hold:

(i) $s^*\alpha Cl(\phi)=\phi$.

(ii) $s^*\alpha Cl(X)=X$.

If A and B are subsets of X,

(iii) $A\subseteq s^*\alpha Cl(A)$.

(iv) $A\subseteq B \Rightarrow s^*\alpha Cl(A)\subseteq s^*\alpha Cl(B)$.

(v) $s^*\alpha Cl(s^*\alpha Cl(A))=s^*\alpha Cl(A)$.

(vi) $A\subseteq spCl(A)\subseteq s\alpha Cl(A)\subseteq s^*\alpha Cl(A)\subseteq \alpha Cl(A)\subseteq Cl(A)$.

(vii) $A\subseteq spCl(A)\subseteq s\alpha Cl(A)\subseteq s^*\alpha Cl(A)\subseteq s^*Cl(A)\subseteq Cl(A)$.

(viii) $A\subseteq spCl(A)\subseteq sCl(A)\subseteq s^*\alpha Cl(A)\subseteq \alpha Cl(A)\subseteq Cl(A)$.

(ix) $A\subseteq spCl(A)\subseteq s^*pCl(A)\subseteq s^*\alpha Cl(A)\subseteq \alpha Cl(A)\subseteq Cl(A)$.

(x) $s^*\alpha Cl(A\cup B)\supseteq s^*\alpha Cl(A)\cup s^*\alpha Cl(B)$.

(xi) $s^*\alpha Cl(A\cap B)\subseteq s^*\alpha Cl(A)\cap s^*\alpha Cl(B)$.

(xii) $Cl(s^*\alpha Cl(A))=Cl(A)$.

(xiii) $s^*\alpha Cl(Cl(A))=Cl(A)$.

(xiv) If (X, τ) is an α-structure and $A\subseteq X$, then $s^*Cl(A)=s^*\alpha Cl(A)$.

(xv) If (X, τ) is extremally disconnected and $A\subseteq X$, then $s^*pCl(A)=s^*\alpha Cl(A)$.

(xvi) If (X, τ) is a locally indiscrete space and $A\subseteq X$, then $Cl(A)=\alpha Cl(A)=s^*Cl(A)=s^*\alpha Cl(A)=s\alpha Cl(A)= sCl(A)$

Proof: (i), (ii), (iii) and (iv) follow from Definition 4.1. From Theorem 4.2 $s^*\alpha Cl(A)$ is semi*α-closed and from Theorem 4.3 $s^*\alpha Cl(s^*\alpha Cl(A))=s^*\alpha Cl(A)$.This proves (v). Clearly (vi), (vii), (viii) and (ix) follow from Theorem 3.12. Now (x) and (xi) follow from (iv) and set theoretic properties. From (iii), we have $A\subseteq s^*\alpha Cl(A)$ and hence $Cl(A)\subseteq Cl(s^*\alpha Cl(A))$. Also from (vi), we have $Cl(A)\supseteq s^*\alpha Cl(A)$ and hence $Cl(A)\supseteq Cl(s^*\alpha Cl(A))$. Therefore $Cl(s^*\alpha Cl(A))=Cl(A)$. This proves (xii). Now statement (xiii) follows from the fact that $Cl(A)$ is closed and hence by Theorem 3.12(ii), $Cl(A)$ is semi*α-closed and by Theorem 4.3, $s^*\alpha Cl(Cl(A))=Cl(A)$.

Remark 4.5. In Theorem 4.4, the inclusions may be strict and equality may also hold.

Theorem 4.6. Let $A\subseteq X$ and let $x\in X$. Then $x\in s^*\alpha Cl(A)$ if and only if every semi*α-open set in X containing x intersects A.

Proof: Let us prove the contra positive of the theorem. Suppose $x\notin s^*\alpha Cl(A)$. Then $x\in X\backslash s^*\alpha Cl(A)=X\backslash\cap\{F: A\subseteq F$ and F is semi*α-closed in $X\}=\cup\{X\backslash F: A\subseteq F$ and F is semi*α-closed in $X\}$ and so $x\in X\backslash F$ for some semi*α-closed set F containing A. Hence $X\backslash F$ is a semi*α-open set containing x that does not intersect A. On the other hand, suppose there is a semi*α-open set U containing x that is disjoint from A. Then $X\backslash U$ is a semi*α-closed set containing A. Therefore by Definition 4.1 $s^*\alpha Cl(A)\subseteq X\backslash U$. Hence $x\notin s^*\alpha Cl(A)$. Thus $x\notin s^*\alpha Cl(A)$ if and only if there is a semi*α-open set containing x that does not intersect A. This proves the theorem.

Theorem 4.7. If A is a subset of X, then

(i) $s^*\alpha Cl(X\backslash A)=X\backslash s^*\alpha Int(A)$.

(ii) $s^*\alpha Int(X\backslash A)=X\backslash s^*\alpha Cl(A)$.

Proof: (i) Let $x\in X\backslash s^*\alpha Int(A)$.Then $x\notin s^*\alpha Int(A)$.This implies that x does not belong to any semi*α-open subset of A. Let F be a semi*α-closed set containing $X\setminus A$. Then $X\setminus F$ is a semi*α-open set contained in A. Therefore $x\notin X\setminus F$ and so $x\in F$. Thus x belongs to every semi*α-closed set containing $X\setminus A$. Hence $x\in s^*\alpha Cl(X\setminus A)$. Therefore $X\setminus s^*\alpha Int(A)\subseteq s^*\alpha Cl(X\setminus A)$. On the other hand, let $x\in s^*\alpha Cl(X\setminus A)$. Then x does not belong to any semi*α-open subset of A. On the contrary, suppose there exists a semi*α-open subset U of A containing x. Then $X\setminus U$ is a semi*α-closed set that contains $X\setminus A$ but not x. This contradiction proves that $x\notin s^*\alpha Int(A)$. Thatis, $x\in X\setminus s^*\alpha Int(A)$. Hence $s^*\alpha Cl(X\setminus A)\subseteq X\setminus s^*\alpha Int(A)$. This proves (i). Clearly statement (ii) can be proved from (i) by replacing A by $X\setminus A$ and taking the complements on both sides.

Theorem 4.8. Let D be a subset of a topological space (X,τ). Then the following statements are equivalent:

i) D is dense in (X, τ^α).
ii) $s^*\alpha Cl(D)=X$.
iii) If F is a semi*α-closed subset of X containing D, then $F=X$
iv) For each $x\in X$, every semi*α-open set containing x intersects D.
v) $s^*\alpha Int(X\setminus D)=\phi$.

Proof: (i)⟹(ii): Let D be dense in (X, τ^α) and let $x\in X$. Let A be a semi*α-open set in X containing x. Then by Definition 2.3, there exists an α-open set U such that $U\subseteq A\subseteq Cl^*(U)$. Since D is dense in (X, τ^α), $U\cap D\neq\phi$ and hence $A\cap D\neq\phi$. Thus every semi*α-open set in X containing x intersects D. Therefore by Theorem 4.6, $x\in s^*\alpha Cl(D)$. Hence $s^*\alpha Cl(D)=X$.

(ii)⟹(i): Suppose $s^*\alpha Cl(D)=X$. Then by Theorem 4.4(vi), $X=s^*\alpha Cl(D)\subseteq \alpha Cl(D)\subseteq X$. Hence $\alpha Cl(D)=X$. That is, D is dense in (X, τ^α).

(ii)⟹(iii): Let F be a semi*α-closed subset of X containing D. Then $X=s^*\alpha Cl(D)\subseteq s^*\alpha Cl(F)=F$ which implies $F=X$.

(iii)⟹(ii): Since $s^*\alpha Cl(D)$ is a semi*α-closed set containing D, by (iii)$s^*\alpha Cl(D)=X$.

(ii)⟹(iv): Let $x\in X$. Then $x\in s^*\alpha Cl(D)$. By Theorem 4.6, every semi*α-open set containing x intersects D.

(iv)⟹(ii): Let $x\in X$. Then every semi*α-open set containing x intersects D. By Theorem 4.6, $x\in s^*\alpha Cl(D)$. Hence $s^*\alpha Cl(D)=X$.

(iv)⟹(v): If possible, let $x\in s^*\alpha Int(X\setminus D)$. Then $s^*\alpha Int(X\setminus D)$ is a nonempty s*α-open set containing x. From (iv), $D\cap(X\setminus D)\neq\phi$. This contradiction proves (v).

(v)⟹(ii): Suppose $s^*\alpha Int(X\setminus D)=\phi$. By Theorem 4.7(ii), $X\setminus s^*\alpha Cl(D)=\phi$. Hence $s^*\alpha Cl(D)=X$.

Remark 4.9: If (X, τ) is an α-structure, then D is dense in X if and only if $s^*\alpha Cl(D)=X$.

Theorem 4.10. If A is a subset of a topological space X,

(i) $s^*\alpha Cl(A)=A\cup Int^*(\alpha Cl(A))$.
(ii) $s^*\alpha Int(A)=A\cap Cl^*(\alpha Int(A))$.

Proof:(i) Now $Int^*(\alpha Cl(A\cup Int^*(\alpha Cl(A))))=Int^*(\alpha Cl(A)\cup\alpha Cl(Int^*(\alpha Cl(A))))=Int^*(\alpha Cl(A))\subseteq A\cup Int^*(\alpha Cl(A))$. Then by Theorem 3.3, $A\cup Int^*(\alpha Cl(A))$ is a semi*α-closed set containing A. Hence by Theorem 4.2, $s^*\alpha Cl(A)\subseteq A\cup Int^*(\alpha Cl(A))$. Since $s^*\alpha Cl(A)$ is semi*α-closed , by invoking Theorem 3.3, we get $Int^*(\alpha Cl(A))\subseteq Int^*(\alpha Cl(s^*\alpha Cl(A)))\subseteq s^*\alpha Cl(A)$.Therefore $A\cup Int^*(\alpha Cl(A))\subseteq s^*\alpha Cl(A)$. This proves (i). Clearly (ii) follows from (i) by replacing A by $X\setminus A$ and taking the complements on both sides and applying Theorem 4.7.

V. Semi*α-Derived Set

Definition 5.1. Let $A\subseteq X$. An element $x\in X$ is called a ***semi*α-limit point*** of A if every semi*α-open set in X containing x intersects $A\setminus\{x\}$.

Definition 5.2. The set of all semi*α-limit points of A is called the ***semi*α-Derived set*** of A.
It is denoted by $D_{s^*\alpha}(A)$.

Lemma 5.3. If $A\subseteq X$, then $D_{s^*\alpha}(A)\subseteq s^*\alpha Cl(A)$.

Proof: Follows from Definition 5.1 and Theorem 4.6

Theorem 5.4. If $A\subseteq X$, then $s^*\alpha Cl(A)=A\cup D_{s^*\alpha}(A)$.

Proof: By definition, $A\subseteq s^*\alpha Cl(A)$. By Lemma 5.2, we have $D_{s^*\alpha}(A)\subseteq s^*\alpha Cl(A)$. Thus $A\cup D_{s^*\alpha}(A)\subseteq s^*\alpha Cl(A)$. On the other hand, let $x\in s^*\alpha Cl(A)$. If $x\in A$, then $x\in A\cup D_{s^*\alpha}(A)$. Suppose $x\notin A$. We claim that x is a semi*α-limit point of A. Let U be a semi*α-open set containing x. Then by Theorem U intersects A. Since $x\notin A$, U intersects $A\setminus\{x\}$ and hence $x\in D_{s^*\alpha}(A)$. Therefore $s^*\alpha Cl(A)\subseteq A\cup D_{s^*\alpha}(A)$. This proves the theorem.

Corollary 5.5. A subset A of X is semi*α-closed if and only if A contains all its semi*α-limit points. That is, A is semi*α-closed if and only if $D_{s^*\alpha}(A)\subseteq A$.

Proof: By Theorem 5.4, $s^*\alpha Cl(A)=A\cup D_{s^*\alpha}(A)$ and by Theorem 4.3, A is semi*α-closed if and only if $s^*\alpha Cl(A)=A$. Hence A is semi*α-closed if and only if $D_{s^*\alpha}(A)\subseteq A$.

Theorem 5.6. (Properties of Semi*α-Derived Set)

In any topological space (X, τ) the following statements hold.

If A and B are subsets of X,

(i) $A\subseteq B\Longrightarrow D_{s^*\alpha}(A)\subseteq D_{s^*\alpha}(B)$.

(ii) $D_{sp}(A)\subseteq D_{s\alpha}(A)\subseteq D_{s^*\alpha}(A)\subseteq D_{\alpha}(A)\subseteq D(A)$.

(iii) $D_{sp}(A)\subseteq D_{s^*p}(A)\subseteq D_{s^*\alpha}(A)\subseteq D_{\alpha}(A)\subseteq D(A)$.

(iv) $D_{sp}(A)\subseteq D_{s\alpha}(A)\subseteq D_{s^*\alpha}(A)\subseteq D_{s^*}(A)\subseteq D(A)$.

(v) $D_{sp}(A)\subseteq D_{s}(A)\subseteq D_{s^*\alpha}(A)\subseteq D_{\alpha}(A)\subseteq D(A)$.

(vi) $D_{s^*\alpha}(D_{s^*\alpha}(A))\backslash A\subseteq D_{s^*\alpha}(A)$.

(vii) $D_{s^*\alpha}(A\cup B)\supseteq D_{s^*\alpha}(A)\cup D_{s^*\alpha}(B)$.

(viii) $D_{s^*\alpha}(A\cap B)\subseteq D_{s^*\alpha}(A)\cap D_{s^*\alpha}(B)$.

(ix) $D_{s^*\alpha}(A\cup D_{s^*\alpha}(A))\subseteq A\cup D_{s^*\alpha}(A)$.

(x) $s^*\alpha Int(A)=A\backslash D_{s^*\alpha}(X\backslash A)$.

(xi) If (X, τ) is an α-structure, then $D_{s^*\alpha}(A)=D_{s^*}(A)$.

(xii) If (X, τ) is an extremally disconnected space, then $D_{s^*\alpha}(A)=D_{s^*p}(A)$.

(xiii) If (X, τ) is a locally indiscrete space and $A\subseteq X$, then D[A]=αD[A]=s*D[A]=s*αD[A]=sαD[A]= sD[A]

Proof: (i) follows from the definition.

(ii), (iii), (iv) and (v) follow from Theorem 2.12 and from definitions.

(vi) Let $x\in D_{s^*\alpha}(D_{s^*\alpha}(A))\backslash A$ and U be a semi*α-open set containing x. Then $U\cap(D_{s^*\alpha}(A)\backslash\{x\})\neq\phi$. Let $y\in U\cap(D_{s^*\alpha}(A)\backslash\{x\})$. Then $x\neq y\in U$ and $y\in D_{s^*\alpha}(A)$. Hence $U\cap(A\backslash\{y\})\neq\phi$. If $z\in U\cap(A\backslash\{y\})$, then $z\neq x$ and $U\cap(A\backslash\{x\})\neq\phi$ which shows that $x\in D_{s^*\alpha}(A)$. This proves (vi).

(vii) Since $A\subseteq A\cup B$, by (i), $D_{s^*\alpha}(A)\subseteq D_{s^*\alpha}(A\cup B)$. Similarly $D_{s^*\alpha}(B)\subseteq D_{s^*\alpha}(A\cup B)$. Hence $D_{s^*\alpha}(A)\cup D_{s^*\alpha}(B)\subseteq D_{s^*\alpha}(A\cup B)$. This proves (vii).The proof for (viii) is similar.

By Theorem 5.4, $A\cup D_{s^*\alpha}(A)=s^*\alpha Cl(A)$ which is semi*α-closed by Theorem 4.3. From Corollary 5.5, we have $D_{s^*\alpha}(A\cup D_{s^*\alpha}(A))\subseteq A\cup D_{s^*\alpha}(A)$.This proves (ix). Let $x\in A\backslash D_{s^*\alpha}(X\backslash A)$. Then $x\notin D_{s^*\alpha}(X\backslash A)$ which implies there is a semi*α-open set U containing x such that $U\cap((X\backslash A)\backslash\{x\})=\phi$. Since $x\in A$, $U\cap(X\backslash A)=\phi$ which implies $U\subseteq A$. Then $x\in U\subseteq A$ and hence $x\in s^*\alpha Int(A)$. On the other hand if $x\in s^*\alpha Int(A)$ then $s^*\alpha Int(A)$ is a semi*α-open set containing x and contained in A. Hence $s^*\alpha Int(A)\cap(X\backslash A)=\phi$ which implies $x\notin D_{s^*\alpha}(X\backslash A)$. Thus $s^*\alpha Int(A)\subseteq A\backslash D_{s^*\alpha}(X\backslash A)$. This proves (x). The statements (xi), (xii) and (xiii) follow from Remark 2.18.

Remark 5.7. It can be seen that the inclusions in Theorem 5.6 may be strict and equality may hold as well.

VI. Conclusion

The newly defined concept of semi*α-closed sets is shown to be weaker than the concepts of α-closed sets and semi*-closed sets but stronger than the concepts of semi α-closed sets, semi-closed sets and semi*-preclosed sets. Explict expressions for semi*α-closure and semi*α-interior of a subset have been found. The properties of semi*α-border, semi*α-frontier and semi*α-exterior have been investigated.

REFERENCES

[1] N. Levine, "Semi-open sets and Semi-continuity in topological spaces", American Mathematical Monthly, 70 (1)(1963), 36-41.

[2] N. Levine, "Generalized Closed Sets in Topology", Rend. Circ. Mat. Palermo, 19(2) (1970), 89-96.

[3] W. Dunham, "A new closure operator for Non-T_1 topologies", Kyungpook Math. J., 22(1) (1982), 55-60.

[4] O. Njastad, "Some Classes of Nearly Open Sets", Pacific J. Math., 15 (3) (1965), 961-970.

[5] G.B. Navalagi, Definition Bank in General Topology, Topology Atlas (2000)

[6] H.A. Othman, "New types of α-continuous mapping", Master Degree Thesis, Al-Mustansiriya University, IRAQ (2004).

[7] S. Pasunkili Pandian, "A Study on Semi Star-Preopen Sets in Topological Spaces", Ph.D Thesis, Manonmaniam Sundaranar University, Tirunelveli, India, 2013.

[8] A. Robert and S. Pious Missier, "A New Class of sets weaker than α-open sets" (submitted for publication)

[9] A. Robert and S. Pious Missier, "A New Class of Nearly Open Sets", International Journal of Mathematical Archive, 3(7), 2012, 1-8.

[10] A.S, Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, "On Precontinuous and Weak Precontinuous mappings", Proc. Math. Phys. Soc. Egypt 53(1982) 47-53.
[11] D. Andrijevic, "Semi-Preopen Sets", Mat. Vesnik, 38(1986), 24-32.
[12] P. Das, "Note on some applications of semiopen sets", Progre. Math. Soc., Allahabad, 7 (1973), 33- 44.
[13] A. Robert and S. Pious Missier, "On Semi*-closed sets", Asian Journal Of Current Engineering And Maths., 1 (4) (2012), 173-176.
[14] S. Willard, General Topology, Addison–Wesley Publishing Company, Inc, 1970.
[15] E.D. Khalimsky, "Applications of Connected Ordered Topological spaces in Topology", Conference of Math. Departments of Povolsia, 1970.