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On Some Integrals of Products of $\overline{H}$-Functions

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Abstract: *The object of the present paper is to evaluate an integral involving products of three $\overline{H}$-function of different arguments which not only provides us the Laplace transform, Hankel transform ([8],p.3), Meijer's Bessel transform ([8],p.121) and various other integral transforms of the product of two $\overline{H}$-functions but also generalizes the result given earlier by many writers notably by Bailey ([3], p.38), Meijer ([8],p.422) and Slater ([20], p.54(3.7.2).*

I. Introduction

The $\overline{H}$-function occurring in the paper will be defined and represented by Inayat-Hussain [12] as follows:

$$\overline{H}_{P,Q}^{M,N}\left[z\right]=\overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \tag{1.1}$$

where
$$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.2}$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j(j=1,...,p)$ and $b_j(j=1,...,Q)$ are complex parameters, $\alpha_j\geq 0(j=1,...,P),\beta_j\geq 0(j=1,...,Q)$ (not all zero simultaneously) and exponents $A_j(j=1,...,N)$ and $B_j(j=N+1,...,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$-function given by equation (1.1) have been given by (Buschman and Srivastava[6]).

$$\Omega\equiv\sum_{j=1}^{M}\left|\beta_j\right|+\sum_{j=1}^{N}\left|A_j\alpha_j\right|-\sum_{j=M+1}^{Q}\left|\beta_jB_j\right|-\sum_{j=N+1}^{P}\left|\alpha_j\right|>0 \tag{1.3}$$

and $\left|\arg(z)\right|<\frac{1}{2}\pi\Omega$ (1.4)

The behavior of the $\overline{H}$-function for small values of $|z|$ follows easily from a result recently given by (Rathie [15],p.306,eq.(6.9)).

We have

$$\overline{H}_{P,Q}^{M,N}\left[z\right]=0\left(|z|^{\gamma}\right),\gamma=\min_{1\leq j\leq N}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right],|z|\to 0 \tag{1.5}$$

If we take $A_j=1(j=1,2,...,N),B_j=1(j=M+1,...,Q)$ in (1.1), the function $\overline{H}_{P,Q}^{M,N}[.]$ reduces to the Fox's H-function [9].

The following series representation for the $\overline{H}$-function will be required in the sequel (see Rathie, [15]pp.305-306,eq.(6.8)):

$$\overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=$$

$$\frac{\sum_{h=1}^{M}\sum_{r=0}^{\infty}\prod_{\substack{j=1\\j\neq h}}^{M}\Gamma\left(b_j-\beta_j\xi_{h,r}\right)\prod_{j=1}^{N}\left\{\Gamma\left(1-a_j+\alpha_j\xi_{h,r}\right)\right\}^{A_j}(-1)^r z^{\xi_{h,r}}}{\prod_{j=M+1}^{Q}\left\{\Gamma\left(1-b_j+\beta_j\xi_{h,r}\right)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma\left(a_j-\alpha_j\xi_{h,r}\right)r!\beta_h} \quad (1.6)$$

Where

$$\xi_{h,r}=\frac{\left(b_h+r\right)}{\beta_h}.$$

II. The $\overline{H}$ -Function Of Two Variables

The $\overline{H}$ -function of two variables will be defined and represented in the following manner:

$$\overline{H}[x,y]=\overline{H}\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]=\overline{H}^{o,n_1:\,m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,p_1},\left(c_j,\gamma_j;K_j\right)_{1,n_2},\left(c_j,\gamma_j\right)_{n_2+1,p_2},\left(e_j,E_j;R_j\right)_{1,n_3},\left(e_j,E_j\right)_{n_3+1,p_3}\\\left(b_j,\beta_j;B_j\right)_{1,q_1},\left(d_j,\delta_j\right)_{1,m_2},\left(d_j,\delta_j;L_j\right)_{m_2+1,q_2},\left(f_j,F_j\right)_{1,m_3},\left(f_j,F_j;S_j\right)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1\left(\xi,\eta\right)\phi_2(\xi)\phi_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \quad (1.7)$$

Where

$$\phi_1\left(\xi,\eta\right)=\frac{\prod_{j=1}^{n_1}\Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)}{\prod_{j=n_1+1}^{p_1}\Gamma\left(a_j-\alpha_j\xi-A_j\eta\right)\prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j\xi+B_j\eta\right)} \quad (1.8)$$

$$\phi_2\left(\xi\right)=\frac{\prod_{j=1}^{n_2}\left\{\Gamma\left(1-c_j+\gamma_j\xi\right)\right\}^{K_j}\prod_{j=1}^{m_2}\Gamma\left(d_j-\delta_j\xi\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(c_j-\gamma_j\xi\right)\prod_{j=m_2+1}^{q_2}\left\{\Gamma\left(1-d_j+\delta_j\xi\right)\right\}^{L_j}} \quad (1.9)$$

$$\phi_3\left(\eta\right)=\frac{\prod_{j=1}^{n_3}\left\{\Gamma\left(1-e_j+E_j\eta\right)\right\}^{R_j}\prod_{j=1}^{m_3}\Gamma\left(f_j-F_j\eta\right)}{\prod_{j=n_3+1}^{p_3}\Gamma\left(e_j-E_j\eta\right)\prod_{j=m_3+1}^{q_3}\left\{\Gamma\left(1-f_j+F_j\eta\right)\right\}^{S_j}} \quad (1.10)$$

Where x and y are not equal to zero (real or complex), and an empty product is interpreted as unity p_i,q_i,n_i,m_j are non-negative integers such that $0\le n_i\le p_i, o\le m_j\le q_j (i=1,2,3;\, j=2,3)$. All the $a_j(j=1,2,...,p_1), b_j(j=1,2,...,q_1), c_j(j=1,2,...,p_2), d_j(j=1,2,...,q_2)$, $e_j(j=1,2,...,p_3), f_j(j=1,2,...,q_3)$ are complex parameters. $\gamma_j\ge 0(j=1,2,...,p_2), \delta_j\ge 0(j=1,2,...,q_2)$ (not all zero simultaneously), similarly $E_j\ge 0(j=1,2,...,p_3), F_j\ge 0(j=1,2,...,q_3)$ (not all zero simultaneously). The exponents $K_j(j=1,2,...,n_3), L_j(j=m_2+1,...,q_2), R_j(j=1,2,...,n_3), S_j(j=m_3+1,...,q_3)$ can take on non-negative values.

The contour L_1 is in ξ -plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma\left(d_j-\delta_j\xi\right)(j=1,2,...,m_2)$ lie to the right and the poles of $\Gamma\left\{\left(1-c_j+\gamma_j\xi\right)\right\}^{K_j}(j=1,2,...,n_2), \Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)(j=1,2,...,n_1)$ to the left of the contour. For $K_j(j=1,2,...,n_2)$ not an integer, the poles of gamma functions of the numerator in (1.9) are converted to the branch points.

The contour L_2 is in η -plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma\left(f_j - F_j\eta\right)(j=1,2,...,m_3)$ lie to the right and the poles of $\Gamma\left\{\left(1-e_j+E_j\eta\right)\right\}^{R_j}$ $(j=1,2,...,n_3), \Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)(j=1,2,...,n_1)$ to the left of the contour. For $R_j(j=1,2,...,n_3)$ not an integer, the poles of gamma functions of the numerator in (1.10) are converted to the branch points.

The functions defined in (1.7) is an analytic function of x and y , if

$$U=\sum_{j=1}^{p_1}\alpha_j+\sum_{j=1}^{p_2}\gamma_j-\sum_{j=1}^{q_1}\beta_j-\sum_{j=1}^{q_2}\delta_j<0 \tag{1.11}$$

$$V=\sum_{j=1}^{p_1}A_j+\sum_{j=1}^{p_3}E_j-\sum_{j=1}^{q_1}B_j-\sum_{j=1}^{q_3}F_j<0 \tag{1.12}$$

The integral in (1.7) converges under the following set of conditions:

$$\Omega=\sum_{j=1}^{n_1}\alpha_j-\sum_{j=n_1+1}^{p_1}\alpha_j+\sum_{j=1}^{m_2}\delta_j-\sum_{j=m_2+1}^{q_2}\delta_jL_j+\sum_{j=1}^{n_2}\gamma_jK_j-\sum_{j=n_2+1}^{p_2}\gamma_j-\sum_{j=1}^{q_1}\beta_j>0 \tag{1.13}$$

$$\Lambda=\sum_{j=1}^{n_1}A_j-\sum_{j=n_1+1}^{p_1}A_j+\sum_{j=1}^{m_2}F_j-\sum_{j=m_2+1}^{q_2}F_jS_j+\sum_{j=1}^{n_3}E_jR_j-\sum_{j=n_2+1}^{p_3}E_j-\sum_{j=1}^{q_1}B_j>0 \tag{1.14}$$

$$|\arg x|<\frac{1}{2}\Omega\pi,\ |\arg y|<\frac{1}{2}\Lambda\pi \tag{1.15}$$

The behavior of the $\overline{H}$ -function of two variables for small values of $|z|$ follows as:

$$\overline{H}[x,y]=0(|x|^{\alpha}|y|^{\beta}),\max\left\{|x|,|y|\right\}\to 0 \tag{1.16}$$

Where

$$\alpha=\min_{1\le j\le m_2}\left[\operatorname{Re}\left(\frac{d_j}{\delta_j}\right)\right] \qquad \beta=\min_{1\le j\le m_2}\left[\operatorname{Re}\left(\frac{f_j}{F_j}\right)\right] \tag{1.17}$$

For large value of $|z|$,

$$\overline{H}[x,y]=0\left\{|x|^{\alpha'},|y|^{\beta'}\right\},\min\left\{|x|,|y|\right\}\to 0 \tag{1.18}$$

Where

$$\alpha'=\max_{1\le j\le n_2}\operatorname{Re}\left(K_j\frac{c_j-1}{\gamma_j}\right),\ \beta'=\max_{1\le j\le n_3}\operatorname{Re}\left(R_j\frac{e_j-1}{E_j}\right) \tag{1.19}$$

Provided that $U<0$ and $V<0$.

If we take $K_j=1(j=1,2,...,n_2), L_j=1(j=m_2+1,...,q_2), R_j=1(j=1,2,...,n_3), S_j=1(j=m_3+1,...,q_3)$ in (1.7), the $\overline{H}$ -function of two variables reduces to H -function of two variables due to [13].

If we set $n_1=p_1=q_1=0$, the $\overline{H}$ -function of two variables breaks up into a product of two $\overline{H}$ -function of one variable namely

$$\overline{H}{}^{0,0:m_2,n_2:m_3,n_3}_{0,0:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}\text{---}:(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n+1,p_2}:(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ \text{---}:(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2}:(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right]$$

$$=\overline{H}{}^{m_2,n_2}_{p_2,q_2}\left[x\middle|\begin{matrix}(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}\\(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2}\end{matrix}\right]\overline{H}{}^{m_3,n_3}_{p_3,q_3}\left[y\middle|\begin{matrix}(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right] \tag{1.20}$$

If $\lambda>0$, we then obtain

$$\lambda^2\overline{H}{}^{0,n_1:\ m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x^{\lambda}\\y^{\lambda}\end{matrix}\middle|\begin{matrix}(a_j,\lambda\alpha_j;A_j)_{1,p_1},(c_j,\lambda\gamma_j;K_j)_{1,n_2},(c_j,\lambda\gamma_j)_{n_2+1,p_2},(e_j,\lambda E_j;R_j)_{1,n_3},(e_j,\lambda E_j)_{n_3+1,p_3}\\(b_j,\lambda\beta_j;B_j)_{1,q_1},(d_j,\lambda\delta_j)_{1,m_2},(d_j,\lambda\delta_j;L_j)_{m_2+1,q_2},(f_j,\lambda F_j)_{1,m_3},(f_j,\lambda F_j;S_j)_{m_3+1,q_3}\end{matrix}\right]$$

$$=\overline{H}^{0,n_1:\ m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right] \quad (1.21)$$

$$\overline{H}^{0,n_1:\ m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}1/x\\1/y\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$=\overline{H}^{0,n_1:\ m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}(1-b_j,\beta_j;B_j)_{1,q_1},(1-d_j,\delta_j)_{1,m_2},(1-d_j,\delta_j;L_j)_{m_2+1,q_2},(1-f_j,F_j)_{1,m_3},(1-f_j,F_j;S_j)_{m_3+1,q_3}\\(1-a_j,\alpha_j;A_j)_{1,p_1},(1-c_j,\gamma_j;K_j)_{1,n_2},(1-c_j,\gamma_j)_{n_2+1,p_2},(1-e_j,E_j;R_j)_{1,n_3},(1-e_j,E_j)_{n_3+1,p_3}\end{matrix}\right.\right] \quad (1.22)$$

III. Main Results

$$c\int_0^\infty \overline{H}^{m_1,n_1}_{p_1,q_1}\left[ax\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}\\(b_j,\beta_j)_{1,m_1},(b_j,\beta_j;B_j)_{m_1+1,q_1}\end{matrix}\right.\right]\overline{H}^{m_2,n_2}_{p_2,q_2}\left[bx\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}\\(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2}\end{matrix}\right.\right]$$

$$\overline{H}^{0,n_3}_{p_3,q_3}\left[cx\left|\begin{matrix}(-e_j,\phi_j)_{n_1+1,p_1}\\(f_j,\theta_j)_{1,m_1},(1-f_j,\theta_j;F_j)_{m_1+1,q_1}\end{matrix}\right.\right]dx$$

$$=\overline{H}^{m_1,n_1:m_2,n_2:0,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix}\frac{a}{c}\\\frac{b}{c}\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(1-\phi+e_j,\phi_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(\theta-f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right] \quad (2.1)$$

Where $\mathrm{Re}\left[\min\dfrac{b_i}{\beta_i}+\min\dfrac{d_j}{\delta_j}-\min\dfrac{e_k}{\theta_k}+1\right]>0; i=1,2,...,m_1;\ j=1,2,...,m_2; k=1,2,...,m_3$ and

$\lambda_1,\lambda_2,\lambda_3>0, |\arg a|<\dfrac{\pi\lambda_1}{2}, |\arg b|<\dfrac{\pi\lambda_2}{2}, |\arg c|<\dfrac{\pi\lambda_3}{2}$,where

$$\lambda_1=\sum_1^{m_1}\beta_j-\sum_{m_1+1}^{q_1}B_j\beta_j+\sum_1^{n_1}A_j\alpha_j-\sum_{n_1+1}^{p_1}\alpha_j$$

$$\lambda_2=\sum_1^{m_2}\delta_j-\sum_{m_2+1}^{q_2}D_j\delta_j+\sum_1^{n_2}C_j\gamma_j-\sum_{n_2+1}^{p_2}\gamma_j$$

$$\lambda_3=\sum_1^{n_3}E_j\phi_j-\sum_{m_3+1}^{q_3}\theta_j-\sum_{n_3+1}^{p_p}\phi_j$$

Proof: On substituting the value of $\overline{H}^{m_1,n_1}_{p_1,q_1}[ax]$ in terms of Mellin-Barnes integral ([16],p.171)in the integrand of (2.1) and changing the order of integration, the integral transforms into

$$\frac{c}{2\pi i}\int_{-i\infty}^{i\infty}\phi_2(\xi)a^{-\xi}\int_0^\infty \overline{H}^{m_2,n_2}_{p_2,q_2}\left[bx\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}\\(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2}\end{matrix}\right.\right]\overline{H}^{m_3,0}_{p_3,q_3}\left[cx\left|\begin{matrix}(-e_j,\phi_j)_{n_1+1,p_1}\\(f_j,\theta_j)_{1,m_1},(1-f_j,\theta_j;F_j)_{m_1+1,q_1}\end{matrix}\right.\right]dxd\xi$$

The change of the order of integration is readily justified by de la Vallee Poussin's theorem ([2],p.504)in view of the conditions stated earlier.

On evaluating the x -integral by means of ([18],p.1143), it gives us

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\phi_2(\xi)\left(\frac{c}{a}\right)^{\xi}\overline{H}^{m_2+m_3,n_2}_{p_2+p_3,q_2+q_3}\left[\frac{b}{c}\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(-e_j,\phi_j)_{n_1+1,p_1}\\(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2},(f_j,\theta_j)_{1,m_1},(1-f_j,\theta_j;F_j)_{m_1+1,q_1}\end{matrix}\right.\right]d\xi$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\phi_2(\xi)\phi_1(\eta)\frac{\prod_1^{n_3}\Gamma\left(1-\phi_j+e_j+(\xi+\eta)\phi_j\right)}{\prod_{n_3+1}^{p_3}\Gamma\left(\phi_j-e_j-(\xi+\eta)\phi_j\right)\prod_1^{q_3}\Gamma\left(\theta_j-f_j+(\xi+\eta)\theta_j\right)}d\xi d\eta$$

On using (1.7), we arrive at the result (2.1).

IV. Particular cases

(i) If we set $n_3 = q_3 = 2,\, p_3 = 1, \theta_1 = \theta_2 = 1, \phi_1 = 1,\, f_1 = 2 - k + \rho, e_1 = -\rho - m - \frac{1}{2}$

$e_2 = m - \rho - \frac{1}{2}, K_j = L_j = R_j = S_j = 1$, then on using the identity

$$\overline{H}_{1,2}^{2,0}\left[x\left|\begin{matrix}(1-k+\rho,1)\\ \left(\frac{1}{2}+m+\rho,1\right),\left(\frac{1}{2}-m+\rho,1\right)\end{matrix}\right.\right] = e^{-\frac{1}{2}x} x^{\rho}\overline{W}_{k,m}(x) \qquad (3.1)$$

We find that

$$a^{\rho+1}\int_0^{\infty} x^{\rho} e^{-\frac{1}{2}x}\overline{W}_{k,m}(ax)\overline{H}_{p_1,q_1}^{m_1,n_1}\left[bx\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}\\ (b_j,\beta_j)_{1,m_1},(b_j,\beta_j;B_j)_{m_1+1,q_1}\end{matrix}\right.\right]\overline{H}_{p_2,q_2}^{m_2,n_2}\left[cx\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}\\ (d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2}\end{matrix}\right.\right]dx$$

$$= \overline{H}_{p_1,q_1:p_2,q_2;1,2}^{m_1,n_1:m_2,n_2:0,1}\left[\begin{matrix}\frac{a}{c}\\ \frac{b}{c}\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(k-1+\rho,1)\\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},\left(-\rho-m-\frac{1}{2},1\right),\left(-\rho+m-\frac{1}{2},1\right)\end{matrix}\right.\right] \qquad (3.2)$$

Where

$$\operatorname{Re}\left(\rho \pm m + \frac{3}{2} + \min\frac{b_i}{\beta_i} + \min\frac{d_i}{\delta_i}\right) > 0 \text{ for}$$

$$i = 1,2,...,n;\ j = 1,2,...,m_1; \lambda_1, \lambda_2 > 0, \operatorname{Re}(a) > 0, |\arg b| < \frac{1}{2}\pi\lambda_1, |\arg c| < \frac{1}{2}\pi\lambda_2$$

For $k = 0, m = \frac{1}{2}$, (3.2)gives Laplace transform of the product of two $\overline{H}$ -functions:

$$a^{\rho+1}\int_0^{\infty} x^{\rho}\,\overline{H}_{p_1,q_1}^{m_1,n_1}\left[bx\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}\\ (b_j,\beta_j)_{1,m_1},(b_j,\beta_j;B_j)_{m_1+1,q_1}\end{matrix}\right.\right]\overline{H}_{p_2,q_2}^{m_2,n_2}\left[cx\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}\\ (d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2}\end{matrix}\right.\right]dx$$

$$= a^{-\rho-1}\overline{H}_{p_1,q_1:p_2,q_2;1,1}^{m_1,n_1:m_2,n_2:0,1}\left[\begin{matrix}\frac{a}{c}\\ \frac{b}{c}\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(-\rho,1)\\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(-\rho-1,1),(-\rho,1)\end{matrix}\right.\right] \qquad (3.3)$$

Where

$$\operatorname{Re}\left(\rho + 1 + \min\frac{b_i}{\beta_i} + \min\frac{d_i}{\delta_i}\right) > 0 \text{ for}$$

$$i = 1,2,...,n;\ j = 1,2,...,m_1; \lambda_1, \lambda_2 > 0, \operatorname{Re}(a) > 0, |\arg b| < \frac{1}{2}\pi\lambda_1, |\arg c| < \frac{1}{2}\pi\lambda_2 .$$

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