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International Journal Of Modern Engineering Research (IJMER)

Exact Solutions for MHD Flow of a Viscoelastic Fluid with the Fractional Burgers' Model in an Annular Pipe

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Abstract: *This paper presents an analytical study for the magnetohydrodynamic (MHD) flow of a generalized Burgers' fluid in an annular pipe. Closed from solutions for velocity is obtained by using finite Hankel transform and discrete Laplace transform of the sequential fractional derivatives. Finally, the figures are plotted to show the effects of different parameters on the velocity profile.*

Keywords: *Generalized Burgers' fluid, finite Hankel transform, discrete Laplace transform.*

I. Introduction

Fluid non-Newtonian does not show a linear relationship between stress and strain rate and received a lot of attention due to the increasing applications of industrial and technological field, such as polymer solutions, paints, blood and heavy oils. No model alone that can describe the behavior of all fluids non-Newtonian because of complex behavior.

Thus, it was suggested many of the foundational equations of non-Newtonian fluid models. The Burgers' fluid is one of them which cannot be described a typical relation between shear stress and the rate of strain, for this reason many models of constitutive equations have been proposed for these fluids[7,8,11]. Many applications of this type of fluid can be found in [1,2,15,18]. Many of the developments in the theory of viscoelastic flows have been mainly restricted to the formulations of the basics equation and constitutive models [12,16], and many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system and nonlinear control theory [3,14,19].

Recently, Tong and Liu [6] studied the unsteady rotating flows of a non- Newtonian fluid in an annular pipe with Oldroyd- B fluid model. Tong … etc [5] discussed the flow of Oldroyd- B fluid with fractional derivative in an annular pipe. Hyder … etc [17] discussed the flow of a viscoelastic fluid with fractional Burgers' model in an annular pipe. Tong … etc [4] discussed the flow of generalized Burgers' fluid in an annular pipe. Khan [13] investigated the (MHD) flow of generalized Oldroyd- B fluid in a circular pipe. Later on [10] investigated the slip effects on MHD flow of a generalized Oldroyd- B fluid with fractional derivative.

In this paper, our aim is to steady the effects of MHD on the unsteady flow of a viscoelastic fluid with fractional generalized Burgers' fluid model in an annular pipe. The exact solution for velocity distribution is established by using the finite Hankel transform and discrete Laplace transform of the sequential fractional derivatives.

II. Governing Equations

The constitutive equations for an incompressible fractional Burger's fluid given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad (1 + \lambda_1^{\alpha}\tilde{D}_t^{\alpha} + \lambda_2^{\alpha}\tilde{D}_t^{2\alpha})\mathbf{S} = \mu(1 + \lambda_3^{\beta}\tilde{D}_t^{\beta})\mathbf{A} \tag{1}$$

where $\mathbf{T}$ denoted the cauchy stress, $-p\mathbf{I}$ is the indeterminate spherical stress, $\mathbf{S}$ is the extra stress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{T}$ is the first Rivlin- Ericksen tensor with the velocity gradient where $\mathbf{L} = \text{grad}\,\mathbf{V}$, μ is the dynamic viscosity of the fluid, λ_1 and λ_3 $(<\lambda_1)$ are the relaxation and retardation times, respectively, λ_2 is the new material parameter of Burger's fluid, α and β the fractional calculus parameters such that $0 \le \alpha \le \beta \le 1$ and $\tilde{D}_t^{p}$ the upper convected fractional derivative define by

$$\tilde{D}_t^{\alpha}\mathbf{S} = D_t^{\alpha}\mathbf{S} + (\mathbf{V}.\nabla)\mathbf{S} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^{T} \tag{2}$$

$$\tilde{D}_t^{\beta}\mathbf{A} = D_t^{\beta}\mathbf{A} + (\mathbf{V}.\nabla)\mathbf{A} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^{T} \tag{3}$$

in which D_t^{α} and D_t^{β} are the fractional differentiation operators of order α and β based on the Riemann-Liouville definition, defined as

$$D_t^p[f(t)] = \frac{1}{\Gamma(1-p)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^p} d\tau \qquad ,0 \le p \le 1 \tag{4}$$

here $\Gamma(.)$ denotes the Gamma function and

$$D_t^{2p}\mathbf{S} = D_t^p(D_t^p\mathbf{S}) \tag{5}$$

The model reduced to the generalized Oldroyd- B model when $\lambda_2 = 0$ and if, in addition to that, $\alpha = \beta = 1$ the ordinary Oldroyd- B model will be obtained.

We consider the MHD flow of an incompressible generalized Burger's fluid due to an infinite accelerating plate. For unidirectional flow, we assume that the velocity field and shear stress of the form

$$\mathbf{V} = w(r,t)e_z \quad ,\mathbf{S} = \mathbf{S}(r,t) \tag{6}$$

where e_z is the unit vector along z - direction .Substituting equation (6) into (1) and taking account of the initial condition

$$\mathbf{S}(r,0) = 0 \tag{7}$$

we obtain

$$(1+\lambda_1^\alpha \mathrm{D}_t^\alpha + \lambda_2^\alpha \mathrm{D}_t^{2\alpha})\mathrm{S}_{rz} = \mu(1+\lambda_3^\beta \mathrm{D}_t^\beta)\partial_r w(r,t) \tag{8}$$

$$(1+\lambda_1^\alpha \mathrm{D}_t^\alpha + \lambda_2^\alpha \mathrm{D}_t^{2\alpha})\mathrm{S}_{zz} - 2\mathrm{S}_{rz}(\lambda_1^\alpha + \lambda_2^\alpha \mathrm{D}_t^\alpha)\partial_r w(r,t) = -2\mu\lambda_3^\beta(\partial_r w(r,t))^2 \tag{9}$$

where $\mathrm{S}_{rr} = \mathrm{S}_{\theta z} = \mathrm{S}_{r\theta} = \mathrm{S}_{\theta\theta} = 0$. Furthermore, it assumes that the conducting fluid is permeated by an imposed magnetic field $\mathbf{H} = [0, \mathrm{H}_0, 0]$ which acts in the positive θ- direction. In the low- magnetic Reynolds number approximation, the magnetic body force is represented as $\sigma \mathrm{H}_0^2 w$, where σ is the electrical conductivity of the fluid. Then in the presence of a pressure gradient in the z- direction, the equation of motion yields the following scalar equation:

$$\rho\frac{dw}{dt} = -\frac{\partial \mathrm{p}}{\partial \mathrm{z}} + \frac{1}{r}\frac{\partial}{\partial r}(\mathrm{r}\,\mathrm{S}_{rz}) - \sigma \mathrm{H}_0^2 w \tag{10}$$

where ρ is the constant density of the fluid. Eliminating S_{rz} between Eqs. (8) and (10), we obtain the following fractional differential equation

$$\begin{aligned}(1+\lambda_1^\alpha \mathrm{D}_t^\alpha + \lambda_2^\alpha \mathrm{D}_t^{2\alpha})\frac{\partial w}{\partial \mathrm{t}} &= -\frac{1}{\rho}(1+\lambda_1^\alpha \mathrm{D}_t^\alpha + \lambda_2^\alpha \mathrm{D}_t^{2\alpha})\frac{\mathrm{dp}}{\mathrm{dz}} + \\ &+ v(1+\lambda_3^\beta \mathrm{D}_t^\beta)(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})w - \mathrm{M}(1+\lambda_1^\alpha \mathrm{D}_t^\alpha + \lambda_2^\alpha \mathrm{D}_t^{2\alpha})w\end{aligned} \tag{11}$$

where $v = \dfrac{\mu}{\rho}$ is the kinematic viscosity and $\mathbf{M} = \dfrac{\sigma \mathrm{H}_0^2}{\rho}$ is the magnetic dimensionless number.

III. Plane Poiseuille Flow

Consider that the flow problem of an incompressible generalized Burgers' fluid is initially at rest between two infinitely long coaxial cylinders of radii R_0 and R_1 $(> R_0)$. At time $t = 0^+$ the fluid is generated due to a constant pressure gradient that acts on the liquid in the z- direction . Referring to Eq. (11), the corresponding fractional partial differential equation that describe such flow takes the form

$$\begin{aligned}(1+\lambda_1^\alpha \mathrm{D}_t^\alpha + \lambda_2^\alpha \mathrm{D}_t^{2\alpha})\frac{\partial w}{\partial \mathrm{t}} &= -\mathrm{A}\left(1+\lambda_1^\alpha \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \lambda_2^\alpha \frac{t^{-2\alpha}}{\Gamma(1-2\alpha)}\right) + \\ &+ v(1+\lambda_3^\beta \mathrm{D}_t^\beta)(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})w - \mathrm{M}(1+\lambda_1^\alpha \mathrm{D}_t^\alpha + \lambda_2^\alpha \mathrm{D}_t^{2\alpha})w\end{aligned} \tag{12}$$

where $\mathrm{A} = \dfrac{1}{\rho}\dfrac{dp}{dx}$ is the constant pressure gradient

The associated initial and boundary conditions are as follows

$$w(r,0) = \frac{\partial}{\partial t}w(r,0) = \frac{\partial^2}{\partial t^2}w(r,0) = 0 \qquad , R_0 \le r \le R_1 \tag{13}$$

$$w(R_0,t) = w(R_1,t) = 0 \qquad ,t > 0 \tag{14}$$

To obtain the exact analytical solution of the above problem (12)- (14), we first apply Laplace transform principle [9] with respect to t, we get

$$s(1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha})\overline{w}=-\frac{A}{s}(1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha})+ \\ +\nu(1+\lambda_3^\beta s^\beta)(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r})\overline{w}-M(1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha})\overline{w} \tag{15}$$

$$\overline{w}(r,0)=0 \\ \overline{w}(R_0,s)=\overline{w}(R_1,s)=0 \qquad ,t>0 \tag{16}$$

where $\overline{w}(r,s)$ is the image function of $w(r,t)$ and s is a transform parameter.

We use the finite Hankel transform [9], defined as follows

$$\overline{w}_H=\int_{R_0}^{R_1} r\,\overline{w}\,B_0(rk_i)dr \qquad ,\ i=1,2,3,\ldots \tag{17}$$

and its inverse is

$$\overline{w}=\frac{\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2\overline{w}_H\,B_0(rk_i)J_0^2(R_1k_i)}{J_0^2(R_0k_i)-J_0^2(R_1k_i)} \tag{18}$$

where k_i are the positive roots of equation $B_0(R_1k_i)=0$ and $B_0(rk_i)=J_0(rk_i)Y_0(R_0k_i)-Y_0(rk_i)J_0(R_0k_i)$ where $J_0(.)$ and $Y_0(.)$ are the Bessel functions of the first and second kinds of order zero.

Now applying finite Hankel transform to Eqs. (15)-(16) with respect to r, we get

$$\overline{w}_H=-\frac{A}{s}\frac{(1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha})}{(s+M)(1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha})+\nu\,k_i^2(1+\lambda_3^\beta s^\beta)} \tag{19}$$

Now, writing Eq. (19) in series form as

$$\overline{w}_H=-A(1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha})\sum_{k=0}^{\infty}(-1)^k\sum_{a+b+c+d+n=k}^{a,b,c,d,n\geq 0}k!M^{c+d}(\nu\,k_i^2)^n(\lambda_1^\alpha)^{-k-1+d} \\ (\lambda_2^\alpha)^{b+c}(\lambda_3^\beta)^n\frac{s^\delta}{a!b!c!d!n!\left(s^{\alpha+1}+\frac{\nu\,k_i^2}{\lambda_1^\alpha}+\frac{M}{\lambda_1^\alpha}\right)^{k+1}} \tag{20}$$

where $\delta=k-1+2\alpha(b+c)-(c+d+n)-\alpha(d+n)+\beta n-\alpha n$. And its discrete inverse Laplace transform [20] will take the form

$$w_H=-A\sum_{k=0}^{\infty}(-1)^k\sum_{a+b+c+d+n=k}^{a,b,c,d,n\geq 0}\frac{M^{c+d}(\nu\,k_i^2)^n(\lambda_1^\alpha)^{-k-1+d}(\lambda_2^\alpha)^{b+c}(\lambda_3^\beta)^n}{a!b!c!d!n!}t^{(\alpha+1)k+(\alpha+1-\delta)-1} \\ \left\{E_{\alpha+1,\alpha+1-\delta}^k\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2+M)t^{\alpha+1}\right)+\frac{\lambda_1^\alpha}{t^\alpha}E_{\alpha+1,\alpha+1-\delta}^k\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2+M)t^{\alpha+1}\right)+\frac{\lambda_2^\alpha}{t^{2\alpha}}E_{\alpha+1,\alpha+1-\delta}^k\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2+M)t^{\alpha+1}\right)\right\}$$

(21)

where $E_{\alpha,\beta}^m(z)=\sum_{j=0}^{\infty}\frac{(j+m)!\,z!}{j!\Gamma(\alpha j+\alpha m+\beta)}$ is the generalized Mittag- Leffler function [9] and to obtain Eq. (21), the following property of inverse Laplace transform is used [20]

$$L^{-1}\left\{\frac{m!\,s^{\lambda-\mu}}{(s^\lambda\mp c)}\right\}=t^{\lambda m+\mu-1}E_{\lambda,\mu}^m(\pm ct^\lambda) \qquad ,\ \left(\operatorname{Re}(s)>|c|^{\frac{1}{\lambda}}\right) \tag{22}$$

Finally, the inverse finite Hankel transform gives the analytic solution of velocity distribution

$$w(r,t)=-\frac{A\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2\,B_0(rk_i)J_0^2(R_1k_i)}{J_0^2(R_0k_i)-J_0^2(R_1k_i)}\left[\sum_{k=0}^{\infty}(-1)^k\sum_{a+b+c+d+n=k}^{a,b,c,d,n\geq 0}\frac{M^{c+d}(\nu\,k_i^2)^n(\lambda_1^\alpha)^{-k-1+d}(\lambda_2^\alpha)^{b+c}(\lambda_3^\beta)^n}{a!b!c!d!n!}t^{(\alpha+1)k+(\alpha+1-\delta)-1}\right. \\ \left.\left\{E_{\alpha+1,\alpha+1-\delta}^k\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2+M)t^{\alpha+1}\right)+\frac{\lambda_1^\alpha}{t^\alpha}E_{\alpha+1,\alpha+1-\delta}^k\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2+M)t^{\alpha+1}\right)+\frac{\lambda_2^\alpha}{t^{2\alpha}}E_{\alpha+1,\alpha+1-\delta}^k\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2+M)t^{\alpha+1}\right)\right\}\right]$$

(23)

3.1 The limiting cases

1- Making the limit of Eq.(23) when $\alpha \neq 0$ and $M \to 0$ (c=d=0) , we can get similar solution velocity distribution for unsteady flows of a viscelastic fluid with the fractional Burgers' model, as obtained in Ref[17]. Thus the velocity field reduces to

$$w(r,t) = -\frac{A\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2\, B_0(rk_i)J_0^2(R_1k_i)}{J_0^2(R_0k_i)-J_0^2(R_1k_i)}\left[\sum_{k=0}^{\infty}(-1)^k \sum_{a+b+n=k}^{a,b,n\geq 0}\frac{(\nu\, k_i^2)^n(\lambda_1^\alpha)^{-k-1}(\lambda_2^\alpha)^{b-n}(\lambda_3^\beta)^n}{a!b!n!}t^{(\alpha+1)k+(\alpha+1-\delta)-1}\right.$$
$$\left.\left\{E_{\alpha+1,\alpha+1-\delta}^{k}\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2)t^{\alpha+1}\right)+\frac{\lambda_1^\alpha}{t^\alpha}E_{\alpha+1,\alpha+1-\delta}^{k}\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2)t^{\alpha+1}\right)+\frac{\lambda_2^\alpha}{t^{2\alpha}}E_{\alpha+1,\alpha+1-\delta}^{k}\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2)t^{\alpha+1}\right)\right\}\right] \quad (24)$$

where $\delta = k-1+2\alpha b-n-2\alpha n+\beta n$.

2- Making the limit of Eq.(23) when $\alpha \neq 0$, $\lambda_2 \to 0$ (b=0) and $M \to 0$ (c=d=0) , we can get the velocity distribution for a generalized Oldroyd- B fluid. Thus the velocity field reduces to

$$w(r,t) = -\frac{A\pi^2}{2}\sum_{i=1}^{\infty}\frac{k_i^2\, B_0(rk_i)J_0^2(R_1k_i)}{J_0^2(R_0k_i)-J_0^2(R_1k_i)}\left[\sum_{k=0}^{\infty}(-1)^k \sum_{a+n=k}^{a,n\geq 0}\frac{(\nu\, k_i^2)^n(\lambda_1^\alpha)^{-k-1}(\lambda_3^\beta)^n}{a!\; n!}t^{(\alpha+1)k+(\alpha+1-\delta)-1}\right.$$
$$\left.\left\{E_{\alpha+1,\alpha+1-\delta}^{k}\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2)t^{\alpha+1}\right)+\frac{\lambda_1^\alpha}{t^\alpha}E_{\alpha+1,\alpha+1-\delta}^{k}\left(-\frac{1}{\lambda_1^\alpha}(\nu k_i^2)t^{\alpha+1}\right)\right\}\right] \quad (25)$$

where $\delta = k-1-n+\beta n$.

IV. Numerical Results And Discussion:

In this work, we have discussed the MHD flow of generalized Burger's fluid in an annular pipe. The exact solution for the velocity field u is obtained by using the discrete Laplace and finite Hankel transforms. Moreover, some figures are plotted to show the behavior of various parameters involved in the expressions of velocity u.

A comparison between the magnetic parameter effect M (Panel a) and no magnetic parameter effect (M=0) (Panel b) is also made graphically in Figs 1-5.

Figs. 1 and 2 provide the graphically illustrations for the effects of the non- integer fractional parameter α and β on the velocity fields. The velocity is increasing with the increased the α and β for both cases ($\mathrm{M}=0\,\&\,\mathrm{M}\neq 0$).

Fig. 3 provides the graphical illustration for the effect of relaxation parameter λ_1 on the velocity fields. The velocity is decreased with the increase of λ_1 for both cases ($\mathrm{M}=0\,\&\,\mathrm{M}\neq 0$).

Figs. 4 and 5 are prepared to show the effect of the material parameter λ_2 and the retardation parameter λ_3 on the velocity field. The velocity is increasing with the increase of λ_2 and λ_3 for the both cases ($\mathrm{M}=0\,\&\,\mathrm{M}\neq 0$).

Fig. 6 is established to show the behavior of the magnetic parameter M for small as well as for long time. It is observed in (Panel a) that for short time $t=0.1$ the increase in magnetic field M will decrease the velocity profile, while quite the opposite effect is observed for long time $t=0.5$ in (Panel b) i.e., the increase in magnetic field M will increase the velocity profile.

Comparison shows that the velocity profile with magnetic field effect is larger when compared with the velocity profile without magnetic field effect. The effect is explained on the long time.

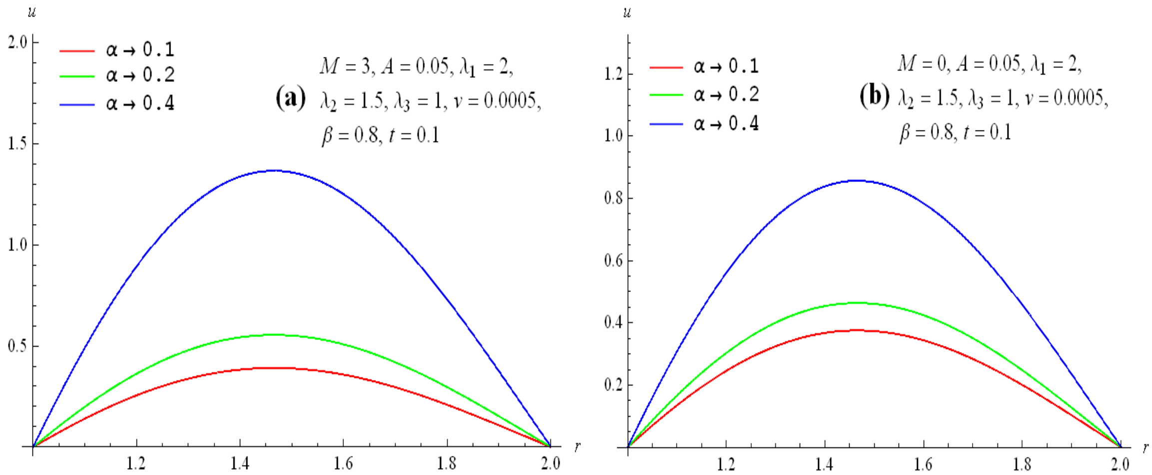

Fig. 1. The velocity for different value of α when keeping other parameters fixed a) $\mathrm{M} = 3$ b) $\mathrm{M} = 0$

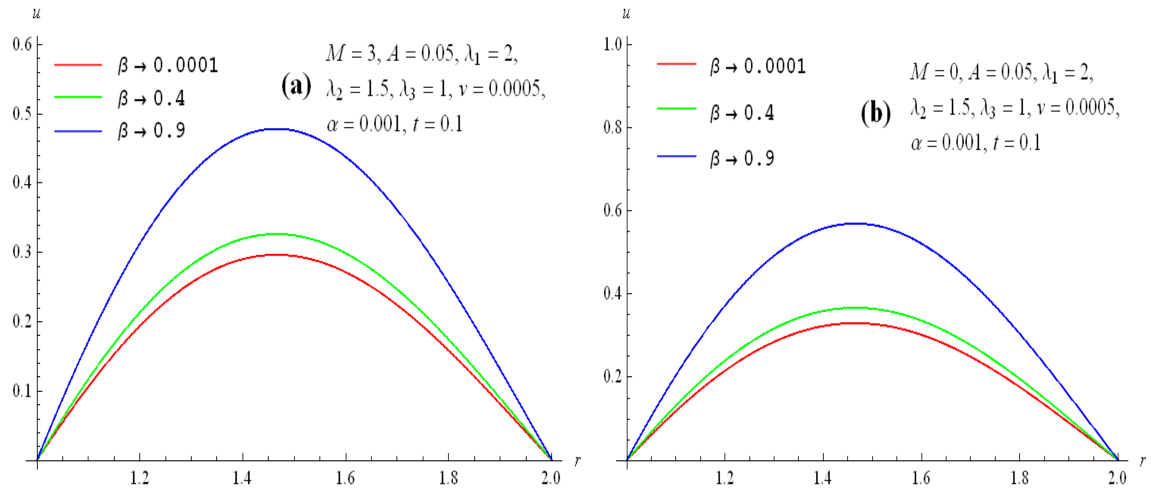

Fig. 2. The velocity for different value of β when keeping other parameters fixed a) $\mathrm{M} = 3$ b) $\mathrm{M} = 0$

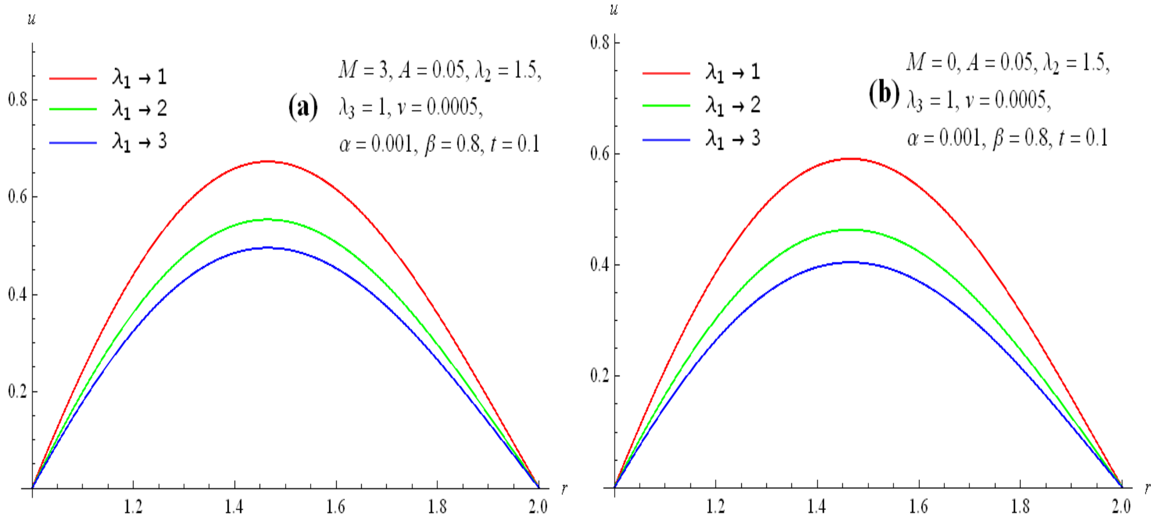

Fig. 3. The velocity for different value of λ_1 when keeping other parameters fixed a) $\mathrm{M} = 3$ b) $\mathrm{M} = 0$

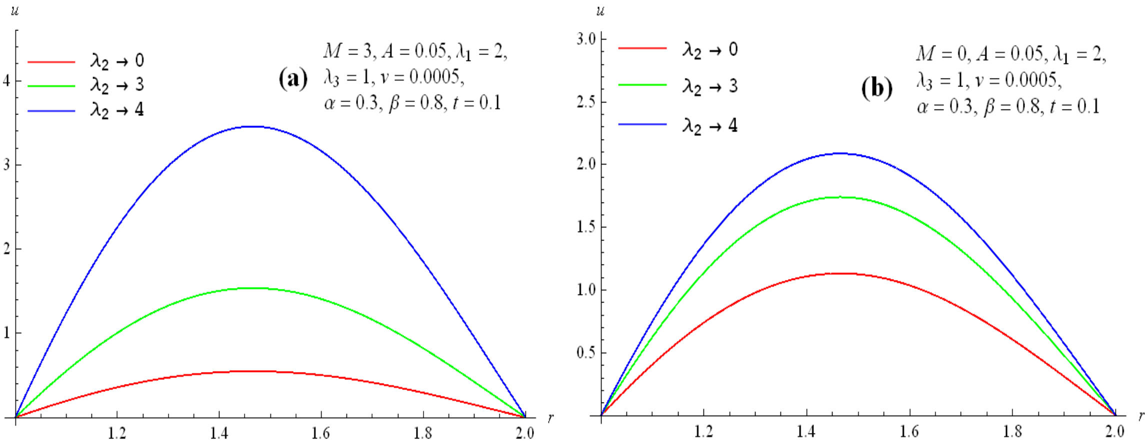

Fig. 4. The velocity for different value of λ_2 when keeping other parameters fixed a) $M = 3$ b) $M = 0$

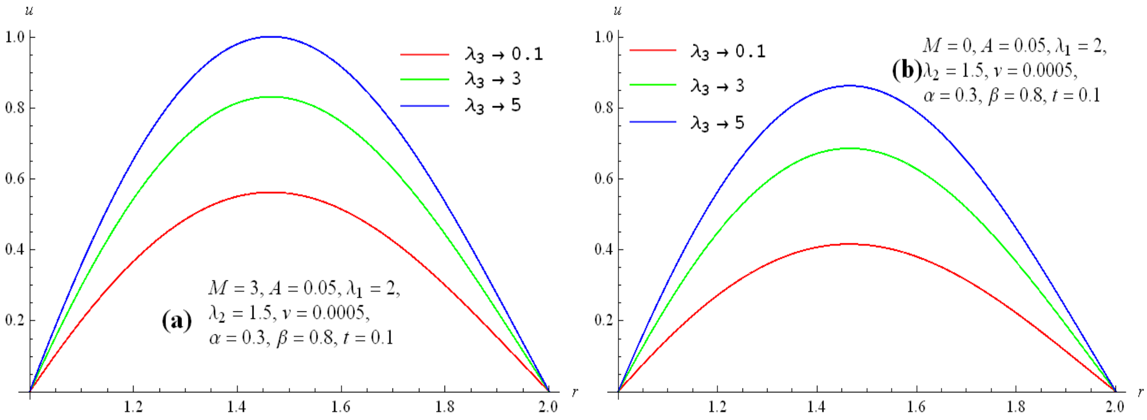

Fig. 5. The velocity for different value of λ_3 when keeping other parameters fixed a) $M = 3$ b) $M = 0$

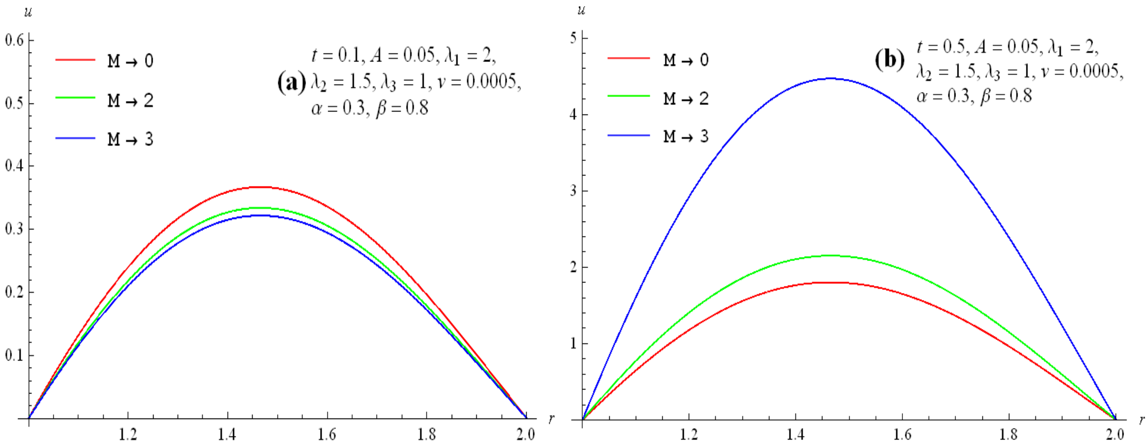

Fig. 6. The velocity for different value of M when keeping other parameters fixed a) $t = 0.1$ b) $t = 0.5$

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