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A method for solving quadratic programming problems having linearly factorized objective function

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Abstract: *A new method namely, objective separable method based on simplex method is proposed for finding an optimal solution to a quadratic programming problem in which the objective function can be factorized into two linear functions. The solution procedure of the proposed method is illustrated with the numerical example.*

Keywords: *Quadratic programming problem, Linear programming problem, Objective separable*

I. Introduction

Non-linear programming problem is an optimization problem in which either the objective function is non-linear, or one or more constraints have non-linear relationship or both. A quadratic programming (QP) problem is the simplest non-linear programming problem in which the objective function is quadratic and the constraints are linear. Because of its usefulness in Management Science, Health Science and Engineering, QP is viewed as a discipline in Operational Research and it has become a fertile area in the field of research in recent years. A large number of algorithms for solving QP problems have been developed. Some of them are extensions of the simplex method and others are based on different principles. In the literature, a great number of methods (Wolfe [1], Beale [2], Frank and Wolfe [3], Shetty [4], Lemke [5], Best and Ritter [6], Theil and van de Panne [7], Boot [8], Fletcher [9], Swarup[10], Gupta and Sharma[11], Moraru [12,13], Jensen and King[14], Bazaraa et al.[15]) are available to solve QP problems. Among them, Wolf's method [1], Swarup's simplex method [10] and Gupta and Sharma's method [11] are more popular than the other methods.

Quazzafi Rabbani and Yusuf Adhami [16] used modified Fourier elimination method for solving QP problems after applying the K-T conditions. Babul Hasan [17] has developed a computer oriented solution method for solving special type of QP problems in which the objective function is a quasi-concave in the feasible region and it can be factorized into two linear functions such that both linear factors are positive in the feasible region. Jan Busa [18] proposed a regularization method for solving the QP problem with linear constraints containing absolute values of variables.

In this paper, we propose a new method namely, objective separable method for finding an optimal solution to a QP problem in which the objective function can be factorized into two linear functions. In this proposed method, we construct two linear programming problems both of maximization type from the given QP problem and then, we obtain an optimal solution to the given QP problem from the solutions of the two constructed linear programming problems. The objective separable method is based only on the simplex method which differs from the existing methods (Wolfe's method [1], Beale's algorithm [2], Swarup's simplex method [10] and Babul Hasan method [17]). Numerical example is given for better understanding the solution procedure of the proposed method.

II. Preliminaries

We need the following definition and the result which can be found in Ezio Marchi [19].

DEFINITION 2.1. Let $f_1(x)$ and $f_2(x)$ be two differentiable functions defined on $X \subset R^n$, an n-dimensional Euclidean space. The functions $f_1(x)$ and $f_2(x)$ are said to have the Gonzi property in $X \subset R^n$ if $(f_1(x) - f_1(u))(f_2(x) - f_2(u)) \le 0$, for all $x, u \in X$.

THEOREM 2.1 The product $f_1(x)f_2(x)$ of two linear functions $f_1(x)$ and $f_2(x)$ is concave if and only if the functions $f_1(x)$ and $f_2(x)$ have the Gonzi property.

III. Quadratic Programming Problems

Consider the following general form of a QP problem:

$$\text{Maximize} \quad f(X)$$
$$\text{subject to} \quad AX \le B, \; X \ge 0$$

where $f : R^n \to R$ is a quadratic function on R^n, X is an *n*-dimensional column vector X, A is an (*m* ×*n*) matrix and B is an *m*-dimensional column vector.

In this paper, we consider a special type of QP problem in which the objective function can be factorized into two linear functions. Such QP problem can be represented as follows:

$$\text{(P)} \quad \text{Maximize} \quad Z(X) = \left(C^T X + \alpha\right)\left(D^T X + \beta\right)$$
$$\text{subject to} \quad AX \le B, \; X \ge 0$$

where A is an (*m* ×*n*) matrix, B is an *m*-dimensional column vector, X, C, D are *n*-dimensional column vectors and α, β are real numbers.

Now, we assume that the two functions $\left(C^T X + \alpha\right)$ and $\left(D^T X + \beta\right)$ have the Gonzi property in the feasible set and the set of all feasible solutions to the problem (P) are non-empty and bounded. Thus, by the Theorem 2.1., it is concluded that the problem (P) is a concave non-linear programming problem with linear constraints. This implies that the optimal solution of the problem (P) exists and it occurs at an extreme point of the feasible region.

Now, from the problem (P), two single objective linear programming problems are constructed as follows:

$$(P_1) \quad \text{Maximize} \quad Z_1(X) = C^T X + \alpha$$
$$\text{subject to} \quad AX \le b, \; X \ge 0$$

and

$$(P_2) \quad \text{Maximize} \quad Z_2(X) = D^T X + \beta$$
$$\text{subject to} \quad AX \le b, \; X \ge 0.$$

REMARK 3.1: From the above theorem, we can easily conclude that if both (P_1) and (P_2) are solvable, then (P) is solvable.

Now, we prove the following theorem connecting the optimal solutions of the problem (P), the problem (P_1) and the problem (P_2) which is used in the proposed method.

THEOREM 3.1: Let X_0 be an optimal solution to the problem (P_1). If $\{X_n\}$ is a sequence of basic feasible solutions to the problem (P_2) by simplex method considering the solution X_0 as an initial feasible solution such that $Z(X_k) \le Z(X_{k+1})$ for all $k = 0,1,2,...,n$ and either X_{n+1} is an optimal solution to the problem (P_2) or $Z(X_{n+1}) \ge Z(X_{n+2})$, then X_{n+1} is an optimal solution to the problem (P).

PROOF: It is obvious that X_{n+1} is a feasible solution to the problem (P)

Let V be a feasible solution to the problem (P).

Now, since X_{n+1} is an optimal solution to the problem (P_2), we have

$Z_2(V) \le Z_2(X_{n+1})$.

Case(i): Now, since $Z(X_k) \le Z(X_{k+1})$, for all $k = 0,1,2,...,n$ and X_{n+1} is an optimal solution to the problem (P_2), we can conclude that $Z(X_{n+1}) \ge Z(V)$. Therefore, X_{n+1} is an optimal solution to the problem (P).

Case(ii): Now, since $Z(X_k) \le Z(X_{k+1})$ for all $k = 0,1,2,...,n$ and $Z(X_{n+1}) \ge Z(X_{n+2})$, we can conclude that $Z(X_{n+1}) \ge Z(V)$. Therefore, X_{n+1} is an optimal solution to the problem (P).

Hence the theorem is proved.

IV. Objective Separable Method

Now, we proposed a new method namely, objective separable method for finding an optimal solution to the QP problem.
The proposed method proceeds as follows:

Step 1: Construct two single objective linear programming problems namely, the problem (P_1) and the problem (P_2) from the given problem (P).

Step 2: Compute the optimal solution to the problem (P_1) using the simplex method. Let the optimal solution to the problem (P_1) be $X_\circ$ and the maximum value of $Z_1(X) = Z_1(X_\circ)$.

Step 3: Use the optimal table of the problem (P_1) as an initial simplex table for the problem (P_2), and obtain a sequence of basic feasible solutions to the problem (P_2) by the simplex method.

Step 4: Let $\{X_n\}$ be a sequence of basic feasible solutions to the problem (P_2) obtained by the Step 3. If $Z(X_k) \le Z(X_{k+1})$ for all $k = 0,1,2,...,n$ and X_{n+1} is an optimal solution to the problem (P_2) for some n, stop the computation process and then, go to the Step 5 or Step 6.

Step 5: If $Z(X_k) \le Z(X_{k+1})$ for all $k = 0,1,2,...,n$ and $Z(X_{n+1}) \ge Z(X_{n+2})$, then stop the computation process and then, go to the Step 6.

Step 6: X_{n+1} is an optimal solution to the problem (P) and the maximum value of $Z(X) = Z(X_{n+1})$ by the Theorem 3.1.

REMARK 4.1: The maximum value for (n +1) is less than or equal to the number of the iterations to obtain the optimal solution to the problem (P_2) by the simplex method.

The proposed method for solving the QP problem is illustrated by the following examples.

EXAMPLE 4.1: Consider the following QP problem.

(P) Maximize $Z = (2x_1 + 4x_2 + x_3 + 1)\,(x_1 - x_2 + 2x_3 + 2)$

subject to $x_1 + 3x_2 \le 4;\ 2x_1 + x_2 \le 3;\ x_2 + 4x_3 \le 3;\ x_1, x_2, x_3 \ge 0$.

The following two LP problems can be obtained from the given problem:

(P_1) Maximize $Z_1(X) = (2x_1 + 4x_2 + x_3 + 1)$

subject to $x_1 + 3x_2 \le 4;\ 2x_1 + x_2 \le 3;\ x_2 + 4x_3 \le 3;\ x_1, x_2, x_3 \ge 0$.

and

(P_2) Maximize $Z_2(X) = (x_1 - x_2 + 2x_3 + 2)$

subject to $x_1 + 3x_2 \le 4;\ 2x_1 + x_2 \le 3;\ x_2 + 4x_3 \le 3;\ x_1, x_2, x_3 \ge 0$.

Now, by simplex method, the optimal solution to the problem (P_1) is $x_1 = 1$, $x_2 = 1$, $x_3 = \frac{1}{2}$, Max. $Z_1 = \frac{15}{2}$ and the value of $Z = \frac{45}{2}$.

Now, by the Step 3 of the proposed method, the solution to the problem (P_2) by simplex method is given below:

Iteration	Solution $(x_1, x_2, x_3, s_1, s_2, s_3)$	Value of Z_1	Value of Z_2	Value of $Z = Z_1 Z_2$
0	$X_0 = (1, 1, \frac{1}{2}, 0, 0, 0)$	$\frac{15}{2}$	3	$\frac{45}{2}$
1	$X_1 = (\frac{3}{2}, 0, \frac{3}{4}, \frac{5}{2}, 0, 0)$	$\frac{19}{4}$	5	$\frac{95}{4}$

Since the 1^{st} iteration table is optimal and by the Step 4 of the proposed method, the optimal solution to the given quadratic programming problem is $x_1 = \frac{3}{2}$, $x_2 = 0$, $x_3 = \frac{3}{4}$ and Maximum value of $Z = \frac{95}{4}$.

EXAMPLE 4.2: Consider the following QP problem.

(P) Maximize $Z = (-x_1 + x_2 + 4)(x_1 - x_2 + 2)$

subject to $-x_1 + 2x_2 \le 2$; $x_1 + x_2 \le 4$; $x_1, x_2 \ge 0$.

The following two LP problems can be obtained from the given problem:

(P_1) Maximize $Z_1 = -x_1 + x_2 + 4$

subject to $-x_1 + 2x_2 \le 2$; $x_1 + x_2 \le 4$; $x_1, x_2 \ge 0$.

and

(P_2) Maximize $Z_2 = x_1 - x_2 + 2$

subject to $-x_1 + 2x_2 \le 2$; $x_1 + x_2 \le 4$; $x_1, x_2 \ge 0$.

Now, by simplex method, the optimal solution to the problem (P_1) is $x_1 = 0$, $x_2 = 1$, Max. $Z_1 = 5$ and the value of $Z = 5$.

Now, by the Step 3 of the proposed method, the solution to the problem (P_2) by simplex method is given below:

Iteration	Solution (x_1, x_2, s_1, s_2)	Value of Z_1	Value of Z_2	Value of $Z = Z_1 Z_2$
0	$X_0 = (0, 1, 0, 3)$	5	1	5
1	$X_1 = (2, 2, 0, 0)$	4	2	8
2	$X_2 = (4, 0, 6, 0)$	0	6	0

Since the 2nd iteration table is optimal and by the Step 4 of the proposed method, the optimal solution to the given linear quadratic programming problem is $x_1 = 0$, $x_2 = 0$ and the Maximum value of $Z = 8$.

V. Conclusion

The objective separable method is proposed to solve QP problems in which the objective function can be factorized into two linear functions. Since the proposed method is based on simplex method, without using KKT conditions and complementarity constraints [2]. Also, we can solve such QP problems using the existing LP solvers. Further, the present work can be extended to integer QP problems and fully fuzzy QP problems.

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