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On Some Partial Orderings for Bimatrices

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Abstract: *The usual star, left-star, right-star, plus order, minus order and Lowner ordering have been generalized to bimatrices. Also it is shown that all these orderings are partial orderings in bimatrices. The relationship between star partial order and minus partial order of bimatrices and their squares are examined.*

Keywords: *Star partial order, left-star partial order, right-star partial order, plus order, minus order and lowner order.*

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I. Introduction And Preliminaries

Let $\mathbb{C}_{mxn}$ be the set of mxn complex bimatrices. The symbols $A_B^*, \mathcal{R}(A_B)$ and $r(A_B)$ denote the conjugate transpose, range space and rank subtractivity of $A_B \epsilon\, \mathbb{C}_{mxn}$ respectively. Further, $A_B^\dagger \epsilon\, \mathbb{C}_{nxm}$ stand for the Moore-Penrose inverse of A_B[10], that is the unique bimatrix satisfying the equations,

$$A_B\, A_B^\dagger A_B = A_B,\; A_B^\dagger\, A_B\, A_B^\dagger = A_B^\dagger, A_B\, A_B^\dagger = \left(A_B\, A_B^\dagger\right)^*, A_B^\dagger A_B = \left(A_B^\dagger A_B\right)^* \qquad (1.1)$$

and I_{Bn} be the identity bimatrix of order n. Moreover, $\mathbb{C}_n^{EP}$, $\mathbb{C}_n^{H}$ and $\mathbb{C}_n^{\geq}$denote the subsets of $\mathbb{C}_{nxn}$ consisting of EP, Hermitian, and Hermitian non-negative definite bimatrices respectively.

That is,

$$\mathbb{C}_n^{EP} = \{A_B \epsilon\, \mathbb{C}_{nxn} \;:\, R(A_B) = R(A_B^*) \Longrightarrow R(A_1) = R(A_1^*)\; and\; R(A_2) = R(A_2^*)\}$$

$$\mathbb{C}_n^{H} = \{A_B \epsilon\, \mathbb{C}_{nxn} \;:\, A_B = A_B^* \Longrightarrow A_1 = A_1^*\; and\; A_2 = A_2^* \} \quad \text{and}$$

$$\mathbb{C}_n^{\geq} = \left\{A_B \epsilon\, \mathbb{C}_{nxn} \;:\, A_B = L_B\, L_B^* \Longrightarrow A_1 = L_1 L_1^*\,;\;\; A_2 = L_2 L_2^*\;\; for\; some L_B = L_1 \cup L_2 \in \mathbb{C}_{nxp}\right\}$$

In this paper, the usual star, left-star, right-star, plus order, minus order and Lowner order have been generalized to bimatrices. Also it is shown that all these orderings are partial orderings in bimatrices. The relationship between star partial order and minus partial order of bimatrices and their squares are examined.

Definition 1.1

The star ordering for bimatrices is defined by,

$$\left.\begin{aligned} A_B \leq^* B_B \Leftrightarrow\; & A_B^* A_B = A_B^* B_B \text{ that is, } A_1^* A_1 = A_1^* B_1\;\; ; A_2^* A_2 = A_2^* B_2 \\ & \text{and } A_B A_B^* = B_B A_B^* \text{ that is, } A_1 A_1^* = B_1 A_1^*\;\; ; A_2 A_2^* = B_2 A_2^* \end{aligned}\right\} \qquad (1.2)$$

and can alternatively be specified as,

$$\left.\begin{aligned} A_B \leq^* B_B \Leftrightarrow\; & A_B^\dagger A_B = A_B^\dagger B_B \text{ that is, } A_1^\dagger A_1 = A_1^\dagger B_1\; ;\; A_2^\dagger A_2 = A_2^\dagger B_2 \\ & \text{and } A_B A_B^\dagger = B_B A_B^\dagger \text{ that is, } A_1 A_1^\dagger = B_1 A_1^\dagger\;\; ;\; A_2 A_2^\dagger = B_2 A_2^\dagger \end{aligned}\right\} \qquad (1.3)$$

Example 1.2

Consider the bimatrices

$$A_B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$$

and $B_B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cup \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$

$\Longrightarrow A_B \leq^* B_B$

Definition 1.3

The left - star ordering for bimatrices is defined by,

$$\left.\begin{array}{l} A_B \;{*}{\leq}\; B_B \Leftrightarrow A_B^* A_B = A_B^* B_B \text{ that is, } A_1^* A_1 = A_1^* B_1 \;;\; A_2^* A_2 = A_2^* B_2 \\ \text{and } \mathcal{R}(A_B) \subseteq \mathcal{R}(B_B) \text{ that is, } \mathcal{R}(A_1) \subseteq \mathcal{R}(B_1) \;;\; \mathcal{R}(A_2) \subseteq \mathcal{R}(B_2) \end{array}\right\} \quad (1.4)$$

Definition 1.4

The right - star ordering for bimatrices is defined by,

$$\left.\begin{array}{l} A_B \leq * B_B \Leftrightarrow A_B A_B^* = B_B A_B^* \text{ that is, } A_1 A_1^* = B_1 A_1^* \;;\; A_2 A_2^* = B_2 A_2^* \\ \text{and } \mathcal{R}(A_B^*) \subseteq \mathcal{R}(B_B^*) \text{ that is, } \mathcal{R}(A_1^*) \subseteq \mathcal{R}(B_1^*) \;;\; \mathcal{R}(A_2^*) \subseteq \mathcal{R}(B_2^*) \end{array}\right\} \quad (1.5)$$

Definition 1.5

The plus - order for bimatrices is defined as, $A_B < B_B$ whenever $A_B^\dagger A_B = A_B^\dagger B_B$ and $A_B A_B^\dagger = B_B A_B^\dagger$, for some reflexive generalized inverse $A_B^\dagger$ of A_B. (satisfying both $A_B A_B^\dagger A_B = A_B$ and $A_B^\dagger A_B A_B^\dagger = A_B^\dagger$).

Definition 1.6

The minus (rank Subtractivity) ordering is defined for bimatrices as,

$$A_B \leq^- B_B \Leftrightarrow \mathrm{r}(B_B - A_B) = \mathrm{r}(B_B) - \mathrm{r}(A_B) \text{ that is, } \mathrm{r}(B_1 - A_1) = \mathrm{r}(B_1) - \mathrm{r}(A_1) \text{ and}$$
$$\mathrm{r}(B_2 - A_2) = \mathrm{r}(B_2) - \mathrm{r}(A_2) \quad (1.6)$$

or as,

$$A_B \leq^- B_B \Leftrightarrow \quad A_B B_B^\dagger B_B = A_B,\; B_B B_B^\dagger A_B = A_B \text{ and } A_B B_B^\dagger A_B = A_B \quad (1.7)$$

Note 1.7

i) It can be shown that, $A_B < B_B \Leftrightarrow \mathrm{r}(B_B - A_B) = \mathrm{r}(B_B) - \mathrm{r}(A_B)$ and so the plus order is equivalent to rank subtractivity.

ii) From (1.4) and (1.5) it is seen that $A_B \leq * B_B \Leftrightarrow A_B^* \;{*}{\leq}\; B_B^*$

Definition 1.8

The Lowner partial ordering denoted by $\leq_L$, for Which $A_B, B_B \in \mathbb{C}_{nXn}$ is defined by

$A_B \leq_L B_B \Leftrightarrow B_B - A_B \in \mathbb{C}_n^{\geq}$.

Result 1.9

Show that,the relation $*\leq$ is a partial ordering.

Proof

(1) $A_B \;{*}{\leq}\; A_B \Rightarrow A_1 \;{*}{\leq}\; A_1$ and $A_2 \;{*}{\leq}\; A_2$ holds trivially.

(2) If $A_B^* A_B = A_B^* B_B$ and $\mathcal{R}(B_B) \subseteq \mathcal{R}(A_B)$ then

$$\begin{aligned} A_B &= A_1 \cup A_2 \\ &= A_1^{\dagger^*} A_1^* A_1 \cup A_2^{\dagger^*} A_2^* A_2 \\ &= A_1^{\dagger^*} A_1^* B_1 \cup A_2^{\dagger^*} A_2^* B_2 \\ &= \left(A_1 A_1^\dagger\right)^* B_1 \cup \left(A_2 A_2^\dagger\right)^* B_2 \\ &= B_1 \cup B_2 \end{aligned}$$

$A_B = B_B$

(3) If $A_B^* A_B = A_B^* B_B$ and $B_B^* B_B = B_B^* C_B$ hold along with $\mathcal{R}(A_B) \subseteq \mathcal{R}(B_B)$ and $\mathcal{R}(B_B) \subseteq \mathcal{R}(C_B)$,then

$$A_B^* A_B = A_B^* B_B$$
$$= A_1^* B_1 \cup A_2^* B_2$$
$$= A_1^* B_1^{\dagger^*} B_1^* B_1 \cup A_2^* B_2^{\dagger^*} B_2^* B_2$$
$$= A_1^* B_1^{\dagger^*} B_1^* C_1 \cup A_2^* B_2^{\dagger^*} B_2^* C_2$$
$$= \left(B_1 B_1^{\dagger} A_1\right)^* C_1 \cup \left(B_2 B_2^{\dagger} A_2\right)^* C_2$$
$$= A_1^* C_1 \cup A_2^* C_2$$
$$A_B^* A_B = A_B^* C_B \text{ and } \mathcal{R}(A_B) \subseteq \mathcal{R}(C_B)$$

Similarly, it can be verified that all the orderings are partial orderings.

Lemma 1.10 [1]

Let $A, B \in \mathbb{C}_{mxn}$ and let $a = r(A) < r(B) = b$. Then $A \leq^* B$ if and only if there exist $U \in \mathbb{C}_{mxb}$,$V \in \mathbb{C}_{nxb}$ satisfying $U^*U = I_b = V^*V$, for which

$$A = U\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} V^* \text{ and } B = U\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} V^* \tag{1.8}$$

where D_1 and D_2 are positive definite diagonal matrices of degree a and $b - a$, respectively. For $A, B \in \mathbb{C}_n^H$, the matrix U in (1.8) may be replaced by V, but then D_1 and D_2 represent any nonsingular real diagonal matrices.

Lemma 1.11[1]

Let $A, B \in \mathbb{C}_{mxn}$ and let $a = r(A) < r(B) = b$. Then $A \leq^- B$ if only if there exist $U \in \mathbb{C}_{mxb}$, $V \in \mathbb{C}_{nxb}$, satisfying $U^*U = I_b = V^*V$, for which

$$A = U\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} V \text{ and } B = U\begin{pmatrix} D_1 + R\,D_2 S & RD_2 \\ D_2 S & D_2 \end{pmatrix} V^* \tag{1.9}$$

Where D_1 and D_2 are positive definite diagonal matrices of degree a and $b - a$, while $R \in \mathbb{C}_{axb-a}$ and $S \in \mathbb{C}_{b-aXa}$ are arbitrary. For $A, B \in \mathbb{C}_n^H$ the matrices U and S in (1.9) may be replaced by V and R^* respectively, but then D_1 and D_2 represent any nonsingular real diagonal matrices.

Lemma 1.12

Let $A_B, B_B \in \mathbb{C}_n^H$ be star-ordered as $A_B \leq^* B_B$. Then $A_B \leq_L B_B$ if and only if $\gamma(A_B) = \gamma(B_B)$, Where γ (.) denotes the number of negative eigenvalues of a given bimatrix.

Proof

Case (i)

Let $r(A_B) = r(B_B)$ that is, $r(A_1) = r(B_1)$ and $r(A_2) = r(B_2)$.

The result is trivial.

Case (ii)

Let $r(A_B) < r(B_B)$ that is, $r(A_1) < r(B_1)$ and $r(A_2) < r(B_2)$ Lemma (1.1) ensures that if $A_B \leq^* B_B$, then

$$B_B - A_B = (B_1 \cup B_2) - (A_1 \cup A_2)$$
$$= (B_1 - A_1) \cup (B_2 - A_2)$$
$$= \left[U_1\begin{pmatrix} D_{11} & 0 \\ 0 & D_{12} \end{pmatrix} V_1^* - U_1\begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix} V_1^*\right] \cup \left[U_2\begin{pmatrix} D_{21} & 0 \\ 0 & D_{22} \end{pmatrix} V_2^* - U_2\begin{pmatrix} D_{21} & 0 \\ 0 & 0 \end{pmatrix} V_2^*\right]$$

$$= (U_1 \cup U_2)\begin{pmatrix} D_{11} \cup D_{12} & 0 \\ 0 & D_{12} \cup D_{22} \end{pmatrix}(V_1^* \cup V_2^*) - (U_1 \cup U_2)\begin{pmatrix} D_{11} \cup D_{21} & 0 \\ 0 & 0 \end{pmatrix}(V_1^* \cup V_2^*)$$

$$= U_B \begin{pmatrix} D_{B1} & 0 \\ 0 & D_{B2} \end{pmatrix} V_B^* - U_B \begin{pmatrix} D_{B1} & 0 \\ 0 & 0 \end{pmatrix} V_B^*$$

$$B_B - A_B = U_B \begin{pmatrix} 0 & 0 \\ 0 & D_{B2} \end{pmatrix} V_B^*$$

Hence it is seen that the order $A_B \leq^L B_B$ is equivalent to the non-negative definiteness of D_{B2}, that is, $\gamma(D_2) = 0$.Consequently, the result follows by noting that $\gamma(A_B) = \gamma(D_{B1})$ and $\gamma(B_B) = \gamma(D_{B1}) + \gamma(D_{B2})$.

II. Star Partial Ordering

Theorem (3) of Baksalary and Pukel sheim [3] asserts that, for any A,B $\in \mathbb{C}_n^{\geq}$,

$$A \leq^* B \Leftrightarrow A^2 \leq^* B^2 \Rightarrow A B = B A \qquad (2.1)$$

This result is revisited here with the emphasis laid on the question which from among four implications comprised in (2.1) continues to be valid for bimatrices not necessarily being bihermitian non negative definite.

Theorem2.1

Let $A_B \in \mathbb{C}_n^{EP}$ and $B_B \in \mathbb{C}_{nXn}$. Then $A_B \leq^* B_B \Rightarrow A_B^2 \leq^* B_B^2$ and $A_B B_B = B_B A_B$ (2.2)

Proof

Since $A_B = A_1 \cup A_2 \in \mathbb{C}_n^{EP} \Leftrightarrow A_B A_B^\dagger = A_B^\dagger A_B$

That is, $A_1 A_1^\dagger = A_1^\dagger A_1$ and $A_2 A_2^\dagger = A_2^\dagger A_2$

Now, $A_B^2 A_B^\dagger = A_1^2 A_1^\dagger \cup A_2^2 A_2^\dagger$

$=A_1(A_1 A_1^\dagger) \cup A_2(A_2 A_2^\dagger)$

$=A_1 \cup A_2$

$A_B^2 A_B^\dagger = A_B$

Also, $A_B^\dagger A_B^2 = A_1^\dagger A_1^2 \cup A_1^\dagger A_1^2$

$=(A_1^\dagger A_1)A_1 \cup (A_2^\dagger)A_2$

$=A_1 \cup A_2$

$A_B^\dagger A_B^2 = A_B$

$\Rightarrow A_B^2 A_B^\dagger = A_B^\dagger A_B^2$

And $(A_B^2)^\dagger = (A_1^2 \cup A_2^2)^\dagger$

$=(A_1 A_1)^\dagger \cup (A_2 A_2)^\dagger$

$=A_1^\dagger A_1^\dagger \cup A_2^\dagger A_2^\dagger$

$=(A_1^\dagger \cup A_2^\dagger)^2$

$(A_B^2)^\dagger = (A_B^\dagger)^2$

Consequently, in view of (1.3),

$A_B B_B = A_1 B_1 \cup A_2 B_2$

$= A_1^2 A_1^\dagger B_1 \cup A_2^2 A_2^\dagger B_2$

$=A_1^2 A_1^\dagger A_1 \cup A_2^2 A_2^\dagger A_2$

$A_B B_B = A_B^2$

And $B_B A_B = B_1 A_1 \cup B_2 A_2$

$= B_1 A_1^{\dagger} A_1^2 \cup B_2 A_2^{\dagger} A_2^2$

$= B_1 A_1^{\dagger} A_1^2 \cup A_2 A_2^{\dagger} A_2^2$

$B_B A_B = A_B^2$

$\Rightarrow A_B B_B = B_B A_B = A_B^2$

Moreover,

$(A_B^2)^{\dagger} B_B^2 = (A_1^2 \cup A_2^2)^{\dagger} (B_1^2 \cup B_2^2)$

$= (A_1^2)^{\dagger} B_1^2 \cup (A_2^2)^{\dagger} B_2^2$

$= (A_1^2)^{\dagger} A_1^2 \cup (A_2^2)^{\dagger} A_2^2$

$= (A_1^2 \cup A_2^2)^{\dagger} (A_1^2 \cup A_2^2)$

$(A_B^2)^{\dagger} B_B^2 = (A_B^2)^{\dagger} A_B^2$

and $B_B^2 (A_B^2)^{\dagger} = B_1^2 (A_1^2)^{\dagger} \cup B_2^2 (A_2^2)^{\dagger}$

$= B_1 A_1 (A_1^{\dagger})^2 \cup B_2 A_2 (A_2^{\dagger})^2$

$= A_1^2 (A_1^2)^{\dagger} \cup A_2^2 (A_2^2)^{\dagger}$

$B_2^2 (A_B^2)^{\dagger} = A_B^2 (A_B^2)^{\dagger}$

$\Longrightarrow A_B^2 \leq^* B_B^2$

Note2.2

Implication 2.2 is not reversible.

Example2.3

Consider the bimatrices,

$A_B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ and $B_B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cup \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$

In which the order $A_B \leq^* B_B$ does not entail either of the conditions $A_B^2 \leq^* B_B^2$, $A_B B_B = B_B A_B$.

On the otherhand, if $A_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cup \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ then $A_B^2 \leq^* B_B^2$, but $A_B^* A_B \neq A_B^* B_B$ and $A_B B_B \neq B_B A_B$.

This showing that even for bihermitian matrices the star order between A_B^2 and B_B^2 and does not entail the star order between A_B and B_B and the commutativity of these bimatrices which are the other two implications contained in (2.1).

It is pointed out that the two conditions on the right-hand side of (2.2) are insufficient for $A_B \leq^* B_B$. When there is no restriction on A_B, a similar conclusion is obtained in the case of combining the two orders $A_B \leq^* B_B$ and ${A_B}^2 \leq^* {B_B}^2$.The bimatrices,

$A_B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $B_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cup \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

and their squares are star ordered, but $A_B B_B \neq B_B A_B$. However, the combination of the order $A_B \leq^* B_B$ with the commutativity condition appears sufficient for $A_B^2 \leq^* B_B^2$ for all quadratic bimatrices.

Theorem2.4

Let $A_B, B_B \in \mathbb{C}_{nxn}$. Then $A_B \leq^* B_B$ and $A_B B_B = B_B A_B \Longrightarrow A_B^2 \leq^* B_B^2$

Proof

On account of (1.2), it follows that if $A_B \leq^* B_B$ and $A_B\ B_B = B_B\ A_B$ then,

$$(A_B^2)^* B_B^2 = (A_1^2 \cup A_2^2)^* (B_1^2 \cup B_2^2)$$
$$= [(A_1^2)^* \cup (A_2^2)^*] (B_1^2 \cup B_2^2)$$
$$= (A_1^2)^* B_1^2 \cup (A_2^2)^* B_2^2$$
$$= A_1^*(A_1^* A_1) B_1 \cup A_2^*(A_2^* A_2) B_2$$
$$= (A_1^*)^2\ A_1\ B_1 \cup (A_2^*)^2 A_2\ B_2$$
$$= A_1^*\ (A_1^*\ B_1) A_1 \cup A_2^*(A_2^* B_2) A_2$$
$$= (A_1^2)^* A_1^2 \cup (A_2^2)^* A_2^2$$
$$(A_B^2)^* B_B^2 = (A_B^2)^* A_B^2$$

and
$$B_B^2 (A_B^2)^* = (B_1^2 \cup B_2^2)\ (A_1^2 \cup A_2^2)^*$$
$$= B_1^2 (A_1^2)^* \cup B_2^2 (A_2^2)^*$$
$$= B_1 (B_1 A_1^*) A_1^* \cup B_2 (B_2 A_2^*) A_2^*$$
$$= B_1 (A_1 A_1^*) A_1^* \cup B_2 (A_2 A_2^*) A_2^*$$
$$= A_1 (B_1 A_1^*) A_1^* \cup A_2 (B_2 A_2^*) A_2^*$$
$$= A_1 (A_1 A_1^*) A_1^* \cup A_2 (A_2 A_2^*) A_2^*$$
$$= A_1^2 (A_1 A_1)^* \cup A_2^2 (A_2 A_2)^*$$
$$= A_1^2 (A_1^2)^* \cup A_2^2 (A_2^2)^*$$
$$B_B^2 (A_B^2)^* = A_B^2 (A_B^2)^*$$

III. Minus Partial Ordering

Baksalary and Pukelsheim [3, Theorem 2] showed that for $,B \in \mathbb{C}_n^{\geq}$, the following three implications hold.

$$A \leq^- B \text{ and } A^2 \leq^- B^2 \implies AB = BA \quad (3.1)$$
$$A \leq^- B \text{ and } AB = BA \implies A^2 \leq^- B^2 \quad (3.2)$$
$$A^2 \leq^- B^2 \text{ and } AB = BA \implies A \leq^- B \quad (3.3)$$

Now,we extend this implications for bimatrices also by the following theorems.

Theorem 3.1

For any $A_B, B_B \in \mathbb{C}_{nxn}$ if $A_B \leq^- B_B$ and $A_B B_B = B_B A_B$ then $A_B^2 \leq^- B_B^2$

Proof

First notice that,

$$A_B \leq^- B_B \text{ and } A_B B_B = B_B A_B \quad (3.4)$$
$$\implies A_B B_B = A_1 B_1 \cup A_2 B_2 = A_1^2 \cup A_2^2 = B_1 A_1 \cup B_2 A_2 = B_B A_B \quad (3.5)$$

On account of 1.7, it follows that,

$$A_B B_B = A_1 B_1 \cup A_2 B_2$$
$$= A_1\ B_1^\dagger\ A_1\ B_1 \cup A_2\ B_2^\dagger\ A_2 B_2$$
$$= A_1\ B_1^\dagger B_1\ A_1 \cup A_2\ B_2^\dagger B_2\ A_2$$
$$= A_1 A_1 \cup A_2 A_2$$

$= A_1^2 \cup A_2^2$

$A_B B_B = A_B^2$

$B_B A_B = B_1 A_1 \cup B_2 A_2$

$= B_1 A_1 B_1^\dagger A_1 \cup B_2 A_2 B_2^\dagger A_2$

$= A_1(B_1 B_1^\dagger A_1) \cup A_2(B_2 B_2^\dagger A_2)$

$= A_1 A_1 \cup A_2 A_2$

$= A_1^2 \cup A_2^2$

$B_B A_B = A_B^2$

$\Longrightarrow A_B B_B = A_B^2 = B_B A_B$

The conditions on the right hand sides of (1.7) and (3.5) lead to the equalities,

$B_B^2 (B_B^2)^\dagger A_B^2 = (B_1^2 \cup B_2^2)(B_1^2 \cup B_2^2)^\dagger (A_1^2 \cup A_2^2)$

$= B_1^2 (B_1^2)^\dagger A_1^2 \cup B_2^2 (B_2^2)^\dagger A_2^2$

$= B_1^2 (B_1^2)^\dagger B_1^2 B_1^\dagger A_1 \cup B_2^2 (B_2^2)^\dagger B_2^2 B_2^\dagger A_2$

$= B_1^2 B_1^\dagger A_1 \cup B_2^2 B_2^\dagger A_2$

$= B_1 A_1 \cup B_2 A_2$

$= B_B A_B$

$\Rightarrow B_B^2 (B_B^2)^\dagger A_B^2 = A_B^2$

$A_B^2 (B_B^2)^\dagger B_B^2 = A_1^2 (B_1^2)^\dagger B_1^2 \cup A_2^2 (B_2^2)^\dagger B_2^2$

$= A_1 B_1 (B_1^2)^\dagger B_1^2 \cup A_2 B_2 (B_2^2)^\dagger B_2^2$

$= A_1 B_1^\dagger B_1^2 (B_1^2)^\dagger B_1^2 \cup A_2 B_2^\dagger B_2^2 (B_2^2)^\dagger B_2^2$

$= A_1 B_1^\dagger B_1^2 \cup A_2 B_2^\dagger B_2^2$

$= A_1 B_1 \cup A_2 B_2$

$= A_B B_B$

$\Rightarrow A_B^2 (B_B^2)^\dagger B_B^2 = A_B^2$

(iii) $A_B^2 (B_B^2)^\dagger A_B^2 = A_1^2 (B_1^2)^\dagger A_1^2 \cup A_2^2 (B_2^2)^\dagger B_2^2$

$= A_1 B_1 (B_1^2)^\dagger B_1 A_1 \cup A_2 B_2 (B_2^2)^\dagger B_2 A_2$

$= A_1 B_1^\dagger B_1^2 B_1^\dagger A_1 \cup A_2 B_2^\dagger B_2^2 B_2^\dagger A_2$

$= A_1 (B_1^\dagger B_1)(B_1 B_1^\dagger) A_1 \cup A_2 (B_2^\dagger B_2)(B_2 B_2^\dagger) A_2$

$= A_1^2 \cup A_2^2$

$A_B^2 (B_B^2)^\dagger A_B^2 = A_B^2$

According to (1.7), which shows that

$A_B^2 \leq^- B_B^2$

Lemma 3.2

Let $A_B \epsilon \mathbb{C}_n^{EP}$.If the Moore – Penrose inverse $B_B^\dagger$ of some $B_B \epsilon \mathbb{C}_n^H$ is a generalized inverse of a bimatrix A_B, that is, $A_B B_{\boldsymbol{B}}^\dagger A_B = A_B$, Then $A_B \epsilon \mathbb{C}_n^H$.

Proof

It is mentioned that, $A_B \epsilon \mathbb{C}_n^{EP}$ if and only if , $A_B A_{\boldsymbol{B}}^{\dagger} = A_{\boldsymbol{B}}^{\dagger} A_B$

Then, on the other hand

$$A_B A_{\boldsymbol{B}}^{\dagger} B_{\boldsymbol{B}}^{\dagger} A_{\boldsymbol{B}}^{*} = A_1 A_1^{\dagger} B_1 A_{\boldsymbol{1}}^{*} \cup A_2 A_2^{\dagger} B_2 A_{\boldsymbol{2}}^{*}$$

$$= (A_1 B_1 A_1 A_1^{\dagger})^* \cup (A_2 B_2 A_2 A_2^{\dagger})^*$$

$$= (A_1 A_1^{\dagger})^* \cup (A_2 A_2^{\dagger})^*$$

$$= A_1 A_1^{\dagger} \cup A_2 A_2^{\dagger}$$

$$A_B A_{\boldsymbol{B}}^{\dagger} B_{\boldsymbol{B}}^{\dagger} A_{\boldsymbol{B}}^{*} = A_{\boldsymbol{B}}^{\dagger} A_B \quad (3.6)$$

Further, $A_B A_{\boldsymbol{B}}^{\dagger} B_{\boldsymbol{B}}^{\dagger} A_{\boldsymbol{B}}^{*} = A_1 A_1^{\dagger} B_1^{\dagger} A_{\boldsymbol{1}}^{*} \cup A_2 A_2^{\dagger} B_2^{\dagger} A_{\boldsymbol{2}}^{*}$

$$= A_1^{\dagger} A_{\boldsymbol{1}}^{*} \cup A_2^{\dagger} A_{\boldsymbol{2}}^{*}$$

$$A_B A_{\boldsymbol{B}}^{\dagger} B_{\boldsymbol{B}}^{\dagger} A_{\boldsymbol{B}}^{*} = A_{\boldsymbol{B}}^{\dagger} A_{\boldsymbol{B}}^{*}$$

Comparing (3.6) with (3.7) we get

$$A_{\boldsymbol{B}}^{\dagger} A_B = A_{\boldsymbol{B}}^{\dagger} A_{\boldsymbol{B}}^{*}$$

Pre-multiplying by A_B,

$$A_B A_{\boldsymbol{B}}^{\dagger} A_B = A_B A_{\boldsymbol{B}}^{\dagger} A_{\boldsymbol{B}}^{*}$$

$$\Rightarrow (A_1 A_1^{\dagger} A_1) \cup (A_2 A_2^{\dagger} A_2) = A_1 A_1^{\dagger} A_{\boldsymbol{1}}^{*} \cup A_2 A_2^{\dagger} A_{\boldsymbol{2}}^{*}$$

$$A_1 \cup A_2 = A_{\boldsymbol{1}}^{*} \cup A_{\boldsymbol{2}}^{*}$$

$$A_B = A_{\boldsymbol{B}}^{*}$$

$$\Rightarrow A_B \epsilon \mathbb{C}_n^{H}$$

Theorem 3.3

Let $A_B \in \mathbb{C}_n^{EP}$ and $B_B \in \mathbb{C}_n^{H}$. Then $A \leq^{-} B$ and $A^2 \leq_L B^2 \Leftrightarrow A_B \leq^{*} B_B$ (3.8)

Proof

Without loss of generality, A_B may be assumed to be Hermitian.

From (1.6) it is clear that and that $r(A_B) \leq r(B_B)$ and that the equality holds only in the trivial case when $A_B = B_B$.

Therefore, assume that $a = r(A_B) < r(B_B) = b$ that is $r(A_1) = r(A_2) = a$ and $r(B_1) = r(B_2) = b$

If $A_B \leq^{*} B_B$, then the fact that $A_{B,} B_B \in \mathbb{C}_n^{H}$ enables representing these bimatrices in the forms described in the second part of Lemma(1.10)**.**

Hence the $\Leftarrow$ part of (3.8) follows.

For the proof of the converse implication observe that, on account of the first two equalities in (1.7)

$$A_B^2 \leq_L B_B^2 \Leftrightarrow B_B^{\dagger} A_B^2 B_B^{\dagger} \leq_L B_B^{\dagger} B_B^2 B_B^{\dagger}$$

$$\Leftrightarrow B_B^{\dagger} A_B^{\dagger} (B_B^{\dagger} A_B)^* \leq_L B_B^{\dagger} B_B \quad (3.9)$$

By conditions (1.1), the Moore-Penrose inverse of a hermitian bimatrix B_B of the form specified in the second part of Lemma (1.11) admits the representation

$$B_B^{\dagger} = V_B \begin{pmatrix} D_{B1}^{-1} & -D_{B1}^{-1} R_B \\ -R_B^* D_{B1}^{-1} & D_{B2}^{-1} + R_B^* D_{B1}^{-1} R_B \end{pmatrix} V_B^*$$

and hence

$$B_B^{\dagger} A_B\left(B_B^{\dagger} A_B\right)^* = V_B \begin{pmatrix} I_{Ba} & 0 \\ -R_B^* & 0 \end{pmatrix} \begin{pmatrix} I_{Ba} & -R_B \\ 0 & 0 \end{pmatrix} V_B^*$$

$$= V_B \begin{pmatrix} I_{Ba} & -R_B \\ -R_B^* & R_B^* R_B \end{pmatrix} V_B^* \qquad (3.10)$$

Since the bimatrix $B_B^{\dagger} B_B (= B_B B_B^{\dagger})$ represents the orthogonal projector onto $\mathcal{R}(B_B) = \mathcal{R}(V_B)$, it may be expressed as $V_B V_B^*$.

Consequently, in view of (3.10)

$$B_B^{\dagger} B_B - B_B^{\dagger} A_B \left(B_B^{\dagger} A_B\right)^* = V_B \begin{pmatrix} 0 & R_B \\ R_B^* & I_{B(b-a)} - R_B^* R_B \end{pmatrix} V_B^* \qquad (3.11)$$

On the account of (3.9), equality (3.11) shows that supplementing the minus order $A_B \leq^- B_B$ by Lowner order $A_B^2 \leq_L B_B^2$ forces to be 0.

Then the bimatrix B_B characterized in lemma (1.11) takes the form described in lemma (1.1), thus leading to the conclusion that

$$A_B \leq^* B_B$$

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