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Between δ-I-closed sets and g-closed sets

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ABSTRACT: *In this paper we introduce a new class of sets known as $\hat{\delta}_S$–closed sets in ideal topological spaces and we studied some of its basic properties and characterizations. This new class of sets lies between δ–I–closed [19] sets and g–closed sets, and its unique feature is it forms topology and it is independent of open sets.*

Keywords and Phrases: *$\hat{\delta}_S$–closed, $\hat{\delta}_S$–open, $\hat{\delta}_S$–closure, $\hat{\delta}_S$–interior.*

I. INTRODUCTION

An ideal I on a topological space (X,τ) is a non empty collection of subsets of X which satisfies (i) A∈I and B⊂A implies B∈I and (ii) A∈I and B∈I implies A∪B∈I. A topological space (X,τ) with an ideal I is called an ideal topological space and is denoted by the triplet (X, τ, I). In an ideal space if P(X) is the set of all subsets of X, a set operator (.)*: P(X) → P(X), called a local function [22] of a with respect to the topology τ and ideal I is defined as follows: for A⊆X, A*(X,τ) = {x∈X / U ∩ A ∉ I, for every U∈τ(x)} where τ(x) = {U∈τ/ x∈U}. A kuratowski closure operator cl*(.) for a topology τ*(I,τ), called the *–topology, finer than τ is defined by cl*(A)=A∪A*(I,τ) [23]. Levine [5], velicko [13], Julian Dontchev and maximilian Ganster [3], Yuksel, Acikgoz and Noiri [14], M.K.R.S. Veerakumar [12] introduced and studied g–closed, δ–closed, δg–closed; δ-I–closed and $\hat{g}$ – closed sets respectively. In 1999, Dontchev [16] introduced Ig–closed sets and Navaneetha Krishnan and Joseph [26] further investigated and characterized Ig–closed sets. The purpose of this paper is to define a new class of closed sets known as $\hat{\delta}_S$–closed sets and also studied some of its basic properties and characterizations.

II. PRELIMINARIES

Definition 2.1. A subset A of a topological space (X, τ) is called a

(i) Semi-open set [10] if A⊆cl(int(A))
(ii) Pre-open set [13] if A⊆int(cl(A))
(iii) α - open set [1] if A⊆int(cl(int(A))
(iv) regular open set[15] if A = int(cl(A))

The complement of a semi-open (resp. pre-open, α - open, regular open) set is called Semi-closed (resp. pre-closed, α - closed, regular closed). The semi-closure (resp. pre closure, α-closure) of a subset A of (X, τ) is the intersection of all semi-closed (resp. pre-closed α-closed) sets containing A and is denoted by scl(A) (resp.pcl(A), αcl(A)). The intersection of all semi-open sets of (X,τ) contains A is called semi-kernel of A and is denoted by sker(A).

Definition 2.2. [18] A subset A of (X,τ) is called δ - closed set in a topological space (X, τ) if A = δcl(A), where $cl_\delta(A)$ = δcl(A) = {x∈X : int(cl(U))∩A≠ϕ, U∈τ(x)}. The complement of δ - closed set in (x, τ) is called δ- open set in (X, τ).

Definition 2.3. [19] Let (X, τ, I) be an ideal topological space, A a subset of X and x a point of X.

(i) x is called a δ-I-cluster point of A if A ∩ int(cl*(U)) ≠ ϕ for each open neighborhood of x.
(ii) The family of all δ-I-cluster points of A is called the δ-I-closure of A and is denoted by $[A]_{\delta\text{–}I}$ and
(iii) A subset A is said to be δ-I-closed if $[A]_{\delta\text{–}I}$ = A. The complement of a δ-I - closed set of X is said to be δ-I - open.

Remark 2.4. From Definition 2.3 it is clear that $[A]_{\delta\text{–}I}$ = {x∈X : int(cl*(U)) ∩ A≠ ϕ, for each U∈τ(x)}

Notation 2.5. Throughout this paper $[A]_{\delta\text{–}I}$ is denoted by σcl(A).

Lemma 2.6. [19] Let A and B be subsets of an ideal topological space (x, τ, I). Then, the following properties hold.

(i) $A \subseteq \sigma cl(A)$
(ii) If $A \subset B$, then $\sigma cl(A) \subset \sigma cl(B)$
(iii) $\sigma cl(A) = \cap\{F \subset X / A \subset F$ and F is δ-I-closed$\}$
(iv) If A is δ-I-closed set of X for each $\alpha \in \Delta$, then $\cap\{A\alpha / \alpha \in \Delta\}$ is δ - I - closed
(v) σcl(A) is δ-I- Closed.

Lemma 2.7. [19] Let (X, τ, I) be an ideal topological space and $\tau_{\delta\text{-}I} = \{A \subset X /A$ is δ-I-open set of (X, τ ,I)$\}$. Then $\tau_{\delta\text{-}I}$ is a topology such that $\tau_s \subset \tau_{\delta\text{-}I} \subset \tau$

Remark 2.8. [19] τ_s (resp. $\tau_{\delta\text{-}I}$) is the topology formed by the family of δ-open sets (resp. δ-I - open sets)

Lemma 2.9. Let (X, τ, I) be an ideal topological space and A a subset of X. Then $\sigma cl(A) = \{x \in X : int(cl^*(U)) \cap A \neq \phi, U \in \tau(x)\}$ is closed.

Proof: If $x \in cl(\sigma l(A))$ and $U \in \tau(x)$, then $U \cap \sigma cl(A) \neq \phi$. Then $y \in U \cap \sigma cl(A)$ for some $y \in X$. Since $U \in \tau(y)$ and $y \in \sigma cl(A)$, from the definition of σcl(A) we have $int(cl^*(U)) \cap A \neq \phi$.Therefore $x \in \sigma cl(A)$, and so $cl(\sigma cl(A)) \subset \sigma cl(A)$ and hence σcl(A) is closed.

Definition 2.10. A subset A of a topological space (X, τ) is called

(i) a generalized closed (briefly g-closed) set [9] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(ii) a generalized semi-closed (briefly sg-closed) set [3] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ).
(iii) a generalized semi-closed (briefly gs-closed) set [2] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(iv) a generalized closed (briefly αg-closed) set [12] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(v) a generalized α-closed (briefly gα-closed) set [12] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α- open in (X, τ).
(vi) a δ– generalized closed (briefly δg -closed) set [4] if $\delta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(vii) a $\hat{g}$ – closed set [17] is $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open.
(viii) a $\delta\hat{g}$ - closed set [8] if $cl_\delta(A) \subseteq U$, whenever $A \subseteq U$ and U is $\hat{g}$ -open set in (X, τ)

The complement of g–closed (resp. sg–closed, gs–closed, αg–closed, gα– closed, δg–closed, $\hat{g}$–closed, $\delta\hat{g}$–closed) set is called g–open (resp.sg – open, gs–open, αg–open, gα– open, δg–open, $\hat{g}$–open, $\delta\hat{g}$–open).

Definition 2.11. A subset A of an ideal space (X, τ, I) is called

(i) Ig-closed set [5] if $A^* \subseteq U$, whenever $A \subseteq U$ and U is open. The complement of Ig-closed set is called Ig-open set.
(ii) R-I-open [19] set if $int(cl^*(A)) = A$. The complement of R- I-open set is R-I-closed

III. $\hat{\delta}_S$ - CLOSED SETS

In this section we introduce the notion of $\hat{\delta}_s$ - closed sets in an ideal topological space (X, τ, I), and investigate their basic properties.

Definition 3.1. A subset A of an ideal topological space (X, τ, I) is called $\hat{\delta}_S$-closed if $\sigma cl(A) \subseteq U$, whenever $A \subseteq U$ and U is semi-open set in (X, τ, I). The complement of $\hat{\delta}_S$-closed set in (X, τ, I), is called $\hat{\delta}_S$-open set in (X, τ, I).

Theorem 3.2. Every δ-closed set is $\hat{\delta}_S$-closed set.

Proof: Let A be any δ-closed set and U be any semi-open set containing A. Since A is δ - closed, $cl_\delta(A) = A$. Always $\sigma cl(A) \subseteq cl_\delta(A)$. Therefore A is $\hat{\delta}_S$-closed set in (X, τ, I).

Remark 3.3. The converse is need not be true as shown in the following example.

Example 3.4. Let X = {a,b,c}, τ = {X, ϕ, {b}, {a,b}, {b,c}, {a,b,c}, {a,b,d}} and I = {ϕ, {c},{d}, {c,d}}. Let A = {a,c,d} then A is $\hat{\delta}_S$-closed but not δ-closed.

Theorem 3.5. Every δ-I – closed set is $\hat{\delta}_S$-closed.

Proof: Let A be any δ - I-closed set and U be any semi-open set such that $A \subseteq U$. Since A is δ-I-closed, σcl(A)=A and hence A is $\hat{\delta}_S$-closed.

Remark 3.6. The following example shows that, the converse is not always true.

Example 3.7. Let X={a,b,c,d}, τ={X, ϕ, {b}, {c}, {b,c}} and I = {ϕ}. Let A = {a,c,d}. Then A is $\hat{\delta}_S$-closed set but not δ-I-closed.

Theorem 3.8. In an ideal topological space (X,τ,I), every $\hat{\delta}_S$-closed set is
(i) $\hat{g}$ - closed set in (X, τ)
(ii) g – closed (resp.gα, αg, sg, gs) – closed set in (X, τ).
(iii) Ig – closed set in (X,τ,I).

Proof. (i) Let A be a $\hat{\delta}_S$-closed set and U be any semi-open set in (X, τ, I) containing A. since A is $\hat{\delta}_S$-closed, σcl(A)⊆U. Then cl(A)⊂U and hence A is $\hat{g}$ - closed in (X, τ).
(ii) By [17], every $\hat{g}$ – closed set is g – closed (resp.gα – closed, αg – closed, sg – closed, gs – closed) set in (X, τ, I). Therefore it holds.
(iii) Since every g – closed set is Ig – closed, It holds.
Remark 3.9. The following example shows that the converse of (i) is not always true.
Example 3.10. Let X = {a,b,c,d}, τ = {X, ϕ,{b},{a,b},{b,c},{a,b,c},{a,b,d}} and I = {ϕ,{b}}. Let A = {c,d}, Then A is $\hat{g}$ – closed set but not $\hat{\delta}_S$-closed.
Remark 3.11. The following examples shows that the converse of (ii) is not true.
Example 3.12. Let X = {a,b,c,d}, τ = {X, ϕ,{c}, {c,d,}} and I = {ϕ,{c},{d},{c,d}}. Let A = {b,d}. Then A is g-closed, αg-closed, gα - closed but not $\hat{\delta}_S$-closed.
Example 3.13. Let X = {a,b,c,d}, τ = {X, ϕ, {a}, {b}, {a,b}, {a,b,d}} and I = {ϕ, {a}, {c}, {a,c}}. Let A = {a,d}. Then A is gs – closed and sg – closed but not $\hat{\delta}_S$-closed.
Remark 3.14. The following example shows that the converse of (iii) is not always true.
Example 3.15. Let X = {a,b,c,d}, τ = {X, ϕ, {a}, {b}, {a,b}} and I = {ϕ}. Let A = {a,b,c}. Then A is Ig - closed set but not $\hat{\delta}_S$-closed.
Remark 3.16. The following examples shows that $\hat{\delta}_S$-closed set is independent of closed, α-closed, semi-closed, δg – closed, δ$\hat{g}$ – closed.
Example 3.17. Let X = {a,b,c,d}, τ = {X, ϕ,{b},{a,b},{b,c},{a,b,c},{a,b,d}} and I = {ϕ,{a},{c},{a,c}}. Let A = {a,d}. Then A is closed, semi-closed but not $\hat{\delta}_S$-closed.
Example 3.18. Let X = {a,b,c,d}, τ = {X, ϕ,{a},{b,c},{a,b,c}} and I = {ϕ,{b}}. Let A = {a,b,d}. Then A is $\hat{\delta}_S$-closed but not closed, semi-closed.
Example 3.19. Let X={a,b,c,d}, τ = {X,ϕ, {a}, {b}, {a,b}} and I={ϕ, {c}}. Let A={b,d}. Then A is δg closed, δ$\hat{g}$ - closed but not $\hat{\delta}_S$-closed.
Example 3.20. Let X = {a,b,c,d}, τ = {X, ϕ, {c}, {a,c}, {b,c}, {a,b,c}, {a,c,d}} and I = {ϕ, {c}}. Let A={a,d}. Then A is $\hat{\delta}_S$-closed but not δg - closed, δ$\hat{g}$ -closed.
Example 3.21. Let X={a,b,c,d}, τ = {X,ϕ, {b},{b,c}} and I = {ϕ}. Let A={c,d}. Then A is α-closed but not $\hat{\delta}_S$-closed.
Example 3.22. Let X={a,b,c,}, τ = {X,ϕ, {b,c}} and I = {ϕ}. Let A={a,c}. Then A is $\hat{\delta}_S$-closed but not α-closed.

IV. CHARACTERIZATIONS

In this section we characterize $\hat{\delta}_S$-closed sets by giving five necessary and sufficient conditions.
Theorem 4.1. Let (X, τ, I) be an ideal space and A a subset of X. Then σcl(A) is semi-closed.
Proof : Since σcl(A) is closed, it is semi-closed.
Theorem 4.2. Let (X, τ, I) be an ideal space and A⊆X. If A⊆B⊆σcl(A) then σcl(A)= σcl(B).
Proof: Since A⊆B, σcl(A) ⊆σcl(B) and since B⊆σcl(A) , then σcl(B) ⊆σcl(σcl(A))= σcl(A), By Lemma 2.6. Therefore σcl(A)= σcl(B).
Theorem 4.3. Let (X, τ, I) be an ideal space and A be a subset of X. then X–σcl(X–A) = σint(A).
Theorem 4.4. Let (X, τ, I) be an ideal topological space. then σcl(A) is always $\hat{\delta}_S$-closed for every subset A of X.
Proof: Let σcl(A)⊆U, where U is semi-open. Always σcl(σcl(A))=σcl(A). Hence σcl(A) is $\hat{\delta}_S$-closed.
Theorem 4.5. Let (X, τ, I) be an ideal space and A⊂X. If sker(A) is $\hat{\delta}_S$-closed then A is also $\hat{\delta}_S$-closed.
Proof: Suppose that sker(A) is a $\hat{\delta}_S$-closed set. If A⊆U and U is semi-open, then sker(A)⊂U. Since sker(A) is $\hat{\delta}_S$-closed σcl(sker(A))⊂U. Always σcl(A)⊂σcl(sker(A)). Thus A is $\hat{\delta}_S$-closed.

Theorem 4.6. If A is $\hat{\delta}_S$-closed subset in (X, τ, I), then σcl(A) – A does not contain any non-empty closed set in (X, τ, I).
Proof**:** Let F be any closed set in (X, τ, I) such that F⊆σcl(A) –A then A⊆X–F and X–F is open and hence semi-open in (X,τ,I). Since A is $\hat{\delta}_S$-closed, σcl(A)⊆X–F. Hence F⊆X–σcl(A). Therefore F⊆(σcl(A) –A) ∩ (X–σcl(A))=ϕ.
Remark 4.7. The converse is not always true as shown in the following example.
Example 4.8. Let X={a,b,c}, τ = {X, ϕ,{a}} and I={ϕ}. Let A={b}. Then σcl(A) –A=X–{b}={a,c} does not contain any non-empty closed set and A is not a $\hat{\delta}_S$-closed subset of (X, τ, I).

Theorem 4.9. Let (X, τ, I) be an ideal space. Then every subset of X is $\hat{\delta}_S$-closed if and only if every semi-open subset of X is δ-I-closed.

Proof**:** Necessity - suppose every subset of X is $\hat{\delta}_S$-closed. If U is semi-open subset of X, then U is $\hat{\delta}_S$-closed and so σcl(U)=U. Hence U is δ-I-closed.
Sufficiency - Suppose A⊂U and U is semi-open. By hypothesis, U is δ-I-closed. Therefore σcl(A)⊂ σcl(U)=U and so A is $\hat{\delta}_S$-closed.

Theorem 4.10**.** Let (X, τ, I) be an ideal space. If every subset of X is $\hat{\delta}_S$-closed then every open subset of X is δ-I-closed.

Proof**:** Suppose every subset of X is $\hat{\delta}_S$-closed. If U is open subset of X, then U is $\hat{\delta}_S$-closed and so σcl(U)=U, since every open set is semi-open. Hence U is δ-I-closed.

Theorem 4.11**.** Intersection of a $\hat{\delta}_S$-closed set and a δ-I-closed set is always $\hat{\delta}_S$-closed.

Proof**:** Let A be a $\hat{\delta}_S$-closed set and G be any δ-I-closed set of an ideal space (X, τ, I). Suppose A∩G⊆U and U is semi-open set in X. Then A⊆U∪(X–G). Now, X–G is δ-I-open and hence open and so semi-open set. Therefore U∪(X–G) is a semi-open set containing A. But A is $\hat{\delta}_S$-closed and therefore σcl(A)⊂ U∪(X–G). Therefore σcl(A)∩G⊂U which implies that, σcl(A∩G)⊂U. Hence A∩G is $\hat{\delta}_S$-closed.

Theorem 4.12. In an Ideal space (X, τ, I), for each x∈X, either{x} is semi-closed or $\{x\}^c$ is $\hat{\delta}_S$-closed set in (X, τ, I).
Proof**:** Suppose that {x} is not a semi-closed set, then $\{x\}^c$ is not a semi-open set and then X is the only semi-open set containing $\{x\}^c$. Therefore σcl($\{x\}^c$) ⊆X and hence $\{x\}^c$ is $\hat{\delta}_S$-closed in (X, τ, I).

Theorem 4.13**.** Every $\hat{\delta}_S$-closed, semi-open set is δ-I-closed.

Proof**:** Let A be a $\hat{\delta}_S$-closed, semi-open set in (X, τ, I). Since A is semi-open, it is semi-open such that A⊆A. Then σcl(A) ⊆A. Thus A is δ-I-closed.

Corollary 4.14. Every $\hat{\delta}_S$-closed and open set is δ-I-closed set.

Theorem 4.15**.** If A and B are $\hat{\delta}_S$-closed sets in an ideal topological space (X, τ, I) then A∪B is $\hat{\delta}_S$-closed set in (X, τ, I).
Proof**:** Suppose that A∪B⊂U, where U is any semi-open set in (X, τ, I). Then A⊆U and B⊆U. Since A and B are $\hat{\delta}_S$-closed sets in (X, τ, I), σcl(A) ⊆U and σcl(B) ⊆U. Always σcl(A∪B)= σcl(A) ∪ σcl(B). Therefore σcl(A∪B) ⊆U for U is semi-open. Hence A is $\hat{\delta}_S$-closed set in (X, τ, I).

Theorem 4.16. Let (X, τ, I) be an ideal space. If A is a $\hat{\delta}_S$-closed subset of X and A⊆B⊆σcl(A), then B is also $\hat{\delta}_S$-closed.

Proof**:** since σcl(B)= σcl(A), by Theorem 4.2. It holds.

Theorem 4.17. A subset A of an ideal space (X, τ, I) is $\hat{\delta}_S$-closed if and only if σcl(A) ⊂ sker(A).

Proof**:** Necessity - suppose A is $\hat{\delta}_S$-closed and x∈σcl(A). If x∉sker(A) then there exists a semi-open set U such that A⊂U but x∉U. Since A is $\hat{\delta}_S$-closed, σcl(A)⊂U and so x∉σcl(A), a contracdiction. Therefore σcl(A) ⊂ sker(A).
Sufficiency - suppose that σcl(A) ⊂ sker(A). If A ⊂ U and U is semi-open. Then sker(A) ⊂ U and so σcl(A) ⊂ U. Therefore A is $\hat{\delta}_S$-closed.
Definition 4.18**.** A subset A of a topological space (X, τ) is said to be a semi∧ - set if sker(A)=A.

Theorem 4.19. Let A be a semi∧ - set of an ideal space (X, τ, I). Then A is $\hat{\delta}_S$-closed if and only if A is δ-I-closed.

Proof: Necessity - suppose A is $\hat{\delta}_S$-closed. Then By Theorem 4.17, σcl(A) ⊂ sker(A) = A, since A is semi∧ -set. Therefore A is δ - I – closed.

Sufficency - The proof follows from the fact that every δ-I-closed set is $\hat{\delta}_S$-closed.

Lemma 4.20. **[6]** Let x be any point in a topological space (X, τ). Then {x} is either nowhere dense or pre-open in (X, τ). Also $X=X_1\cup X_2$, where X_1={x∈X:{x} is nowhere dense in (X, τ)} and X_2={x∈X:{x} is pre-open in (X, τ)} is known as Jankovic – Reilly decomposition.

Theorem 4.21. In an ideal space (X, τ, I), $X_2\cap$σcl(A) ⊆ sker(A) for any subset A of (X, τ, I).

Proof: Suppose that x∈$X_2\cap$σcl(A) and x∉sker(A). Since x∈X_2,{x}⊂int(cl ({x})) and so scl({x}) = int(cl({x})). Since x∈σcl(A), A ∩ int (cl*(U))≠ϕ, for any open set U containing x. Choose U = int(cl({x})), Then A∩int(cl ({x}))≠ϕ. Choose y∈ A∩int(cl({x})). Since x∉sker(A), there exists a semi-open set V such that A ⊆ V and x∉V. If G = X–V, then G is a semi-closed set such that x∈G ⊆ X – A. Also scl({x})=int(cl({x}))⊆G and hence y∈A∩G, a contradiction. Thus x∈sker(A).

Theorem 4.22. A subset A is $\hat{\delta}_S$-closed set in an ideal topological space (X, τ, I). If and only if $X_1\cap$σcl (A)⊆A.

Proof: Necessity - suppose A is $\hat{\delta}_S$-closed set in (X, τ, I) and x ∈ X_1 ∩ σcl(A). Sppose x∉A, then X – {x} is a semi-open set containing A and so σcl(A) ⊆ X – {x}. Which is impossible.

Sufficiency - Suppose X_1 ∩ σcl(A) ⊆ A. Since A ⊆ sker (A), X_1 ∩ σcl(A) ⊆ sker(A). By Theorem 4.21, X_2 ∩ σcl(A) ⊆ sker(A). Therefore σcl(A) = (X_1 ∪ X_2) ∩ σcl(A) = (X_1∩ σcl(A)) ∪ (X_2 ∩ σcl(A)) ⊆ sker(A). By Theorem 4.17, A is $\hat{\delta}_S$-closed in (X, τ, I).

Theorem 4.23. Arbitrary intersection of $\hat{\delta}_S$-closed sets in an ideal space (X, τ, I) is $\hat{\delta}_S$-closed in (X, τ, I).

Proof: Let {A_α : α∈Δ} be any family of $\hat{\delta}_S$-closed sets in (X, τ, I) and A = $\cap_{\alpha\in\Delta} A_\alpha$. Therefore X_1 ∩ σcl(A_α) ⊆ A_α for each α∈Δ and hence X_1 ∩ σcl(A) ⊆ X_1 ∩ σcl(A_α) ⊆ A_α, for each α∈Δ. Then X_1 ∩ σcl(A) ⊆ $\cap_{\alpha\in\Delta} A_\alpha$ = A. By Theorem 4.22, A is $\hat{\delta}_S$-closed set in (X, τ, I). Thus arbitrary intersection of $\hat{\delta}_S$-closed sets in an ideal space (X, τ, I) is $\hat{\delta}_S$-closed in (X, τ, I).

Definition 4.24. A proper non-empty $\hat{\delta}_S$-closed subset A of an ideal space (X, τ, I) is said to be maximal $\hat{\delta}_S$-closed if any $\hat{\delta}_S$-closed set containing A is either X or A.

Examples 4.25. Let X = {a, b, c, d}, τ = {X, ϕ, {c}, {d}, {c, d}} and I = {ϕ}. Then {a, b, c} is maximal $\hat{\delta}_S$-closed set.

Theorem 4.26. In an ideal space (X, τ, I), the following are true

(i) Let F be a maximal $\hat{\delta}_S$-closed set and G be a $\hat{\delta}_S$-closed set. Then F ∪ G = X or G ⊂ F.

(ii) Let F and G be maximal $\hat{\delta}_S$-closed sets. Then F ∪ G = X or F = G.

Proof: (i) Let F be a maximal $\hat{\delta}_S$-closed set and G be a $\hat{\delta}_S$-closed set. If F ∪ G = X. Then there is nothing to prove. Assume that F∪G≠X. Now, F⊂F∪G. By Theorem 4.15, F ∪ G is a $\hat{\delta}_S$-closed set. Since F is maximal $\hat{\delta}_S$-closed set we have, F ∪ G = X or F ∪ G = F. Hence F ∪ G = F and so G ⊂ F.

(ii) Let F and G be maximal $\hat{\delta}_S$-closed sets. If F∪G=X, then there is nothing to prove. Assume that F∪G≠X. Then by (i) F ⊂ G and G ⊂ F which implies that F = G.

Theorem 4.27. A subset A of an ideal space (X,τ ,I) is $\hat{\delta}_S$-open if and only if F⊆σint(A) whenever F is semi-closed and F⊆A.

Proof: Necessity - suppose A is $\hat{\delta}_S$-open and F be a semi-closed set contained in A. Then X–A ⊆ X–F and hence σcl(X–A) ⊂ X – F. Thus F ⊆ X – σcl(X–A) = σint(A).

Sufficiency - suppose X–A ⊆ U, where U is semi-open. Then X–U ⊆ A and X–U is semi-closed. Then X–U ⊆ σint(A) which implies σcl(X –A) ⊆ U. Therefore X–A is $\hat{\delta}_S$-closed and so A is $\hat{\delta}_S$-open.

Theorem 4.28. If A is a $\hat{\delta}_S$-open set of an ideal space (X, τ, I) and σint(A) ⊆ B ⊆ A. Then B is also a $\hat{\delta}_S$-open set of (X, τ, I).

Proof: Suppose F ⊆ B where F is semi-closed set. Then F ⊆ A. Since A is $\hat{\delta}_S$-open, F ⊆ σint(A). Since σint(A) ⊆ σint(B), we have F ⊆ σint(B). By the above Theorem 4.27, B is $\hat{\delta}_S$-open.

V. $\hat{\delta}_S$–CLOSURE

In this section we define $\hat{\delta}_S$-closure of a subset of X and proved it is **"Kuratowski closure operator".**

Definition 5.1. Let A be a subset of an ideal topological space (X, τ, I) then the $\hat{\delta}_S$-closure of A is defined to be the intersection of all $\hat{\delta}_S$-closed sets containing A and it is denoted by $\hat{\delta}_S cl(A)$. That is $\hat{\delta}_S cl(A) = \cap\{F : A \subseteq F$ and F is $\hat{\delta}_S$-closed}. Always $A \subseteq \hat{\delta}_S cl(A)$.

Remark 5.2. From the definition of $\hat{\delta}_S$-closure and Theorem 4.23, $\hat{\delta}_S cl(A)$ is the smallest $\hat{\delta}_S$-closed set containing A.

Theorem 5.3. Let A and B be subsets of an ideal space (X, τ, I). Then the following holds.

(i) $\hat{\delta}_S cl(\phi) = \phi$ and $\hat{\delta}_S cl(X) = X$

(ii) If $A \subseteq B$, then $\hat{\delta}_S cl(A) \subseteq \hat{\delta}_S cl(B)$

(iii) $\hat{\delta}_S cl(A \cup B) = \hat{\delta}_S cl(A) \cup \hat{\delta}_S cl(B)$

(iv) $\hat{\delta}_S cl(A \cap B) \subseteq \hat{\delta}_S cl(A) \cap \hat{\delta}_S cl(B)$

(v) A is a $\hat{\delta}_S$-closed set in (X, τ, I) if and only if $A = \hat{\delta}_S cl(A)$

(vi) $\hat{\delta}_S cl(A) \subseteq \sigma cl(A)$

(vii) $\hat{\delta}_S cl(\hat{\delta}_S cl(A)) = \hat{\delta}_S cl(A)$.

***Proof*:** (i) The proof is obvious.

(ii) $A \subseteq B \subseteq \hat{\delta}_S cl(B)$. But $\hat{\delta}_S cl(A)$ is the smallest $\hat{\delta}_S$-closed set containing A. Hence $\hat{\delta}_S cl(A) \subseteq \hat{\delta}_S cl(B)$.

(iii) $A \subseteq A\cup B$ and $B \subseteq A\cup B$. By (ii) $\hat{\delta}_S cl(A) \subseteq \hat{\delta}_S cl(A\cup B)$ and $\hat{\delta}_S cl(B) \subseteq \hat{\delta}_S cl(A\cup B)$. Hence $\hat{\delta}_S cl(A) \cup \hat{\delta}_S cl(B) \subseteq \hat{\delta}_S cl(A\cup B)$. On the otherhand, $A \subseteq \hat{\delta}_S cl(A)$ and $B \subseteq \hat{\delta}_S cl(B)$ then $A\cup B \subseteq \hat{\delta}_S cl(A) \cup \hat{\delta}_S cl(B)$. But $\hat{\delta}_S cl(A\cup B)$ is the smallest $\hat{\delta}_S$cl-closed set containing $A\cup B$. Hence $\hat{\delta}_S cl(A\cup B) \subseteq \hat{\delta}_S cl(A) \cup \hat{\delta}_S cl\ cl(B)$. Therefore $\hat{\delta}_S cl\ cl(A\cup B) = \hat{\delta}_S cl(A) \cup \hat{\delta}_S cl(B)$.

(iv) $A\cap B \subseteq A$ and $A\cap B \subseteq B$. By (ii) $\hat{\delta}_S cl(A\cap B) \subseteq \hat{\delta}_S cl(A)$ and $\hat{\delta}_S cl(A\cap B) \subset \hat{\delta}_S cl(B)$. Hence $\hat{\delta}_S cl(A\cap B) \subseteq \hat{\delta}_S cl(A) \cap \hat{\delta}_S cl(B)$.

(v) Necessity - Suppose A is $\hat{\delta}_S$-closed set in (X, τ, I). By Remark 5.2, $A \subseteq \hat{\delta}_S cl(A)$. By the definition of $\hat{\delta}_S$-closure and hypothesis $\hat{\delta}_S cl(A) \subseteq A$. Therefore $A = \hat{\delta}_S cl(A)$.

Sufficiency - Suppose $A = \hat{\delta}_S cl(A)$. By the definition of $\hat{\delta}_S$-closure, $\hat{\delta}_S cl(A)$ is a $\hat{\delta}_S$ - closed set and hence A is a $\hat{\delta}_S$-closed set in (X, τ, I).

(vi) Suppose $x \notin \sigma cl(A)$. Then there exists a δ-I-closed set G such that $A \subseteq G$ and $x \notin G$. Since every δ-I-closed set is $\hat{\delta}_S$-closed, $x \notin \hat{\delta}_S cl(A)$. Thus $\hat{\delta}_S cl(A) \subseteq \sigma cl(A)$.

(vii) Since arbitrary intersection of $\hat{\delta}_S$-closed set in an ideal space (X, τ, I) is $\hat{\delta}_S$-closed, $\hat{\delta}_S cl(A)$ is $\hat{\delta}_S$-closed. By (v) $\hat{\delta}_S cl(\hat{\delta}_S cl(A)) = \hat{\delta}_S cl(A)$.

Remark 5.4. The converse of (iv) is not always true as shown in the following example.

Example 5.5. Let X = {a, b, c, d}, τ = {X, ϕ, {a}, {c, d}, {a, c, d}} and I = {ϕ, {d}}. Let A = {c} and B = {d}. Then $\hat{\delta}_S cl(A)$ = {b, c}, $\hat{\delta}_S cl(B)$ = {b, d}. Since $A\cap B = \phi$, $\hat{\delta}_S cl(A\cap B) = \phi$. But $\hat{\delta}_S cl(A) \cap \hat{\delta}_S cl(B)$ = {b}.

Remark 5.6. From $\hat{\delta}_S cl(\phi) = \phi$, $A \subseteq \hat{\delta}_S cl(A)$, $\hat{\delta}_S cl(A\cup B) = \hat{\delta}_S cl(A) \cup \hat{\delta}_S cl(B)$ and $\hat{\delta}_S cl(\hat{\delta}_S cl(A)) = \hat{\delta}_S cl(A)$, we can say that $\hat{\delta}_S$ - closure is the "**Kuratowski Closure Operator**" on (X, τ, I).

Theorem 5.7. In an Ideal space (X, τ, I), for $x \in X$, $x \in \hat{\delta}_S cl(A)$ if and only if $U\cap A \neq \phi$ for every $\hat{\delta}_S$-open set U containing x.

***Proof*:** Necessity - Suppose $x \in \hat{\delta}_S cl(A)$ and suppose there exists a $\hat{\delta}_S$- open set U containing x such that $U\cap A = \phi$. Then $A \subset U^c$ is the $\hat{\delta}_S$-closed set. By Remark 5.2, $\hat{\delta}_S cl(A) \subseteq U^c$. Therefore $x \notin \hat{\delta}_S cl(A)$, a contradiction. Therefore $U\cap A \neq \phi$.

Sufficiency – Suppose $U\cap A\neq\phi$. for every $\hat{\delta}_S$-open set U containing x and suppose $x\notin \hat{\delta}_S cl(A)$. Then there exist a $\hat{\delta}_S$-closed set F containing A such that $x\notin F$. Hence F^c is $\hat{\delta}_S$-open set containing x such that $F^c\subseteq A^c$. Therefore $F^c\cap A=\phi$ which contradicts the hypothesis. Therefore $x\in\hat{\delta}_S cl(A)$.

Theorem 5.8. Let (X, τ, I) be an ideal space and $A\subseteq X$. If $A\subseteq B\subseteq \hat{\delta}_S cl(A)$ Then $\hat{\delta}_S cl(A) = \hat{\delta}_S cl(B)$.

***Proof*:** The Proof is follows from the fact that $\hat{\delta}_S cl(\hat{\delta}_S cl(A)) = \hat{\delta}_S cl(A)$.

Definition 5.9. Let A be a subset of a space (X, τ, I). A point x in an ideal space (X, τ, I) is said to be a $\hat{\delta}_S$-interior point of A. If there exist some $\hat{\delta}_S$-open set U containing x such that $U\subseteq A$. The set of all $\hat{\delta}_S$-interior points of A is called $\hat{\delta}_S$-interior of A and is denoted by $\hat{\delta}_S int(A)$.

Remark 5.10. $\hat{\delta}_S int(A)$ is the Union of all $\hat{\delta}_S$-open sets contained in A and by the Theorem 4.15, $\hat{\delta}_S int(A)$ is the largest $\hat{\delta}_S$-open set contained in A.

Theorem 5.11.

(i) $X - \hat{\delta}_S cl(A) = \hat{\delta}_S int(X - A)$.

(ii) $X - \hat{\delta}_S int(A) = \hat{\delta}_S cl(X - A)$.

(iii)

***Proof*:** (i) $\hat{\delta}_S int(A)\subseteq A\subseteq \hat{\delta}_S cl(A)$. Hence $X-\hat{\delta}_S cl(A)\subseteq X - A\subseteq X-\hat{\delta}_S int(A)$. Then $X - \hat{\delta}_S cl(A)$ is the $\hat{\delta}_S$-open set contained in (X – A). But $\hat{\delta}_S int(X - A)$ is the largest $\hat{\delta}_S$-open set contained in (X–A). Therefore $X-\hat{\delta}_S cl(A)\subseteq\hat{\delta}_S int(X - A)$. On the other hand if $x\in\hat{\delta}_S int(X-A)$ then there exist a $\hat{\delta}_S$-open set U containing x such that $U\subset X-A$ Hence $U\cap A=\phi$. Therefore $x\notin\hat{\delta}_S cl(A)$ and hence $x\in X-\hat{\delta}_S cl(A)$. Thus $\hat{\delta}_S int(X - A)\subseteq X - \hat{\delta}_S cl(A)$.

(ii) Similar to the proof of (i)

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