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Integration by Parts for D_K Integral

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ABSTRACT: *In this paper we have defined D_k integral and proved the integration by parts formula.*

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I. PRELIMINERIES

Let *f* be a real valued function defined on a set *E*. Let $c,d \in E, c<d$, and $k \geq 2$ the oscillation of *f* on $[c,d]\cap E$ of order *k* is defined to be

$$O_k(f,[c,d]\cap E) = \sup|(d-c)[f,c,x_1,x_2,\ldots x_{k-1},d]|$$

where the sup is taken over all points $x_1,x_2,\ldots x_{k-1}$ on $[c,d]\cap E$ and $[f,c,x_1,\ldots,x_{k-1},d]$ represents the k^{th} order divided difference of *f* at the *k*+1 points $c,x_1,\ldots,x_{k-1},d$.

The weak variation of *f* of order *k* is defined as follows,

$$V_k(f,E) = \sup\sum_i O_k(f,[c_i,d_i]\cap E)$$

where the sup is taken over all sequences $\{(c_i,d_i)\}$ of non overlapping intervals with end points on *E*. Then *f* is said to be of k^{th} variation in the wide sense if $V_k(f,E)<\infty$ and it is written as $f \in BV_k(E)$. The function *f* is said to be k^{th} absolutely continuous on *E* if for any $\varepsilon>0$ there is $\delta>0$ such that for every sequence of non overlapping intervals (c_i,d_i) with end points on *E* and with $\sum_i(d_i-c_i)<\delta$ we have $\sum_i O_k(f,[c_i,d_i]\cap E)<\varepsilon$ and we write it as $f \in AC_k(E)$.

The function *F* is said to be generalised absolutely k^{th} continuous (resp. of generalised bounded k^{th} variation) on *E* if $E=\bigcup E_i$ where each E_i is closed and $f \in AC_k(E_i)$ (resp. $f \in BV_k(E_i)$) for each *i* and we write it as $f \in AC_kG(E)$ (resp. $f \in BV_kG(E)$)

II. AUXILIARY RESULT

Following result will be needed which are proved in [2]

Lemma 2.1 *Let E be a closed set. Then* $f \in AC_kG(E)$ *(resp.* $f \in BV_kG(E)$*) if and only if every closed subset of E has a portion on which f is* AC_k*(resp.*BV_k*.)*

Lemma 2.2 *The classes of functions* $AC_k(E), BV_k(E), AC_kG(E), BV_kG(E)$ *are all linear spaces.*

Theorem 2.3 *If* $f \in BV_k(E_i)$ *then* $f_{ap}^{(k)}$ *exists finitely a.e on E, where* $f_{ap}^{(k)}$ *is repeated approximate derivative of f.*

Theorem 2.4 *Let k≥2 and* $f:[a,b]\to R$ *be such that*

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i) $f \in AC_k G([a,b]) \cap D[a,b]$

ii) $f_{ap}^{(k-2)}$ *exists in [a,b]*

iii) if for r=0,1,...,k-2 and $x \in (a,b)$ one of $(f_{ap}^{(r)})_+'(x)$ and $(f_{ap}^{(r)})_-'(x)$ exists then $(f_{ap}^{(r)})'(x)$ exists.(this condition is weaker than the smoothness condition of $f_{ap}^{(r)}$).

iv) $f_{ap}^{(k)} \geq 0$ *a.e*

Then $f^{(k-1)}$ exists and is nondecreasing in [a,b] and. $f^{(k-1)} \in AC_1([a,b])$

The following theorem is proved for *k*=2 in Theorem-4 of [2], and similarly can be proved any *k*.

Theorem 2.5 *Let F and G be AC_k (resp.BV_k) on E . Then FG is AC_k (resp.BV_k) on E. Then FG is AC_k (resp.BV_k) on E .*

Corollary 2.6 *If F and G are $AC_k G$ (resp.$BV_k G$) on [a,b] then FG is so in [a,b]*

The proof is similar of the corollary of Theorem-4 of [2].

III. THE D_k INTEGRAL

A function $f:[a,b] \to R$ is said to be D_k integrable on [*a*,*b*] if there exists a continuous function $\phi:[a,b] \to R$ such that

i) $f \in AC_k G([a,b])$

ii) $\phi_{ap}^{(k-2)}$ exists in [*a*,*b*]

iii) $\phi_{ap}^{(r)}$ is smooth in (*a*,*b*) for *r*=0,1,...,*k*-2.

iv) $\phi_{ap}^{(k-1),+}(a)$, $\phi_{ap}^{(k-1),-}(b)$ exists and $\phi_{ap}^{(k)} = f$ a.e in [*a*,*b*]. Where $\phi_{ap}^{(k-1),+}(a)$ and $\phi_{ap}^{(k-1),-}(b)$ denote right hand approximate derivative of ϕ at *a* of order *k*-1 and left hand approximate derivative of ϕ at *b* of order *k*-1. (The existence of $\phi_{ap}^{(k)}$ a.e in [*a*,*b*] is guaranteed by Theorem-2.3)

The function ϕ if exists is called k^{th} primitive of *f* on [*a*,*b*]. If $F = \phi_{ap}^{(k-1)}$ we call *F* to be an indefinite D_k integral of *f* and

$$\phi_{ap}^{(k-1),-}(b) - \phi_{ap}^{(k-1),+}(a) = F(b) - F(a)$$

is called the definite D_k integral of *f* over [*a*,*b*]and is denoted by

$$(D_k) \int_a^b f(t)dt.$$

The definite D_k integral is unique by Theorem-2.4. The indefinite D_k integral is unique upto an additive constant and the k^{th} primitive is unique upto an addition of polynomial of degree *k*-1.

Theorem 3.1 *Let f be D_k integrable on [a,b] and on [b,c] and let ϕ and ψ be k^{th} primitives of f in [a,b] and [b,c] respectively. If $\phi^{(k-1),-}(b)$ and $\psi^{(k-1),+}(b)$ exists then f is D_k integrable in [a,c] and*

$$(D_k) \int_a^b f + (D_k) \int_b^c f = (D_k) \int_a^c f.$$

Proof: Since $\phi_{ap}^{(k-1),-}(b)$ and $\psi_{ap}^{(k-1),+}(b)$ exists, the previous derivative exists in some left and right neighbourhood of b . Let

$$H(x) = \begin{cases} \phi(x)+\psi(b)+(x-b)\psi^{(1),+}(b)+\ldots+\dfrac{(x-b)^{(k-1)(}}{(k-1)!}(x-b)\psi^{(k-1),+}(b) if, x\in[a,b], \\ \psi(x)+\phi(b)+(x-b)\phi^{(1),-}(b)+\ldots+\dfrac{(x-b)^{(k-1)(}}{(k-1)!}(x-b)\phi^{(k-1),-}(b) if, x\in[b,c]. \end{cases}$$

Then H is continuous in [a,c]. We show that H is the k^th primitive of f in [a,c]. Clearly $H \in AC_kG([a,c])$. Also for $0 \le r \le k-2,$

$$H_{ap}^{(r)}(x) = \begin{cases} \phi_{ap}^{(r)}(x)+\psi^{(r),+}(b)+(x-b)\psi^{(r+1),+}(b)+\ldots+\dfrac{(x-b)^{(k-r-1)}}{(k-r-1)!}\psi^{(k-1),+}(b), if, x\in[a,b), \\ \psi_{ap}^{(r)}(x)+\phi^{(r),-}(b)+(x-b)\phi^{(r+1),-}(b)+\ldots+\dfrac{(x-b)^{(k-r-1)}}{(k-r-1)!}\phi^{(k-1),-}(b), if, x\in(b,c], \\ \psi^{(r),+}(b)+\phi^{(r),-}(b), if, x=b \end{cases}$$

And hence $H_{ap}^{(k-2)}$ exists in [a,c]. We are to show that $H_{ap}^{(r)}$ is smooth at b for $0 \le r \le k-2.$ Let $h > 0$ and $0 \le r \le k-2.$ Then

$$\frac{H_{ap}^{(r)}(b+h)+H_{ap}^{(r)}(b-h)-2H_{ap}^{(r)}(b)}{h} = \frac{H_{ap}^{(r)}(b+h)-H_{ap}^{(r)}(b)}{h} + \frac{H_{ap}^{(r)}(b-h)-H_{ap}^{(r)}(b)}{h}$$

$$= \frac{\psi_{ap}^{(r)}(b+h)-\psi^{(r),+}(b)+h\phi^{(r+1),-}(b)}{h} + \frac{\phi_{ap}^{(r)}(b-h)-\phi^{(r),-}(b)-h\psi^{(r+1),+}(b)}{h} + O(h)$$

The first term tends to $\phi^{(r+1),-}(b)+\psi^{(r+1),+}(b)$ and the second term tends to $-\phi^{(r+1),-}(b)-\psi^{(r+1),+}(b)$ as $h \to 0.$ So $H_{ap}^{(r)}$ is smooth at b for $0 \le r \le k-2.$ Hence $H_{ap}^{(r)}(x)$ is smooth on (a,c) for $0 \le r \le k-2.$ The proof of other properties of k^th primitive are easy. Hence H is the k^th primitive of f on [a,c]. So f is D_k integrable on [a,c]. Also

$$(D_k)\int_a^c f = H_{ap}^{(k-1),-}(c) - H_{ap}^{(k-1),-}(a)$$

$$= \{\phi^{(k-1),-}(b)+\psi^{(k-1),-}(c)\} - \{\phi^{(k-1),+}(a)+\psi^{(k-1),+}(b)\}$$

$$= \{\phi^{(k-1),-}(b)-\phi^{(k-1),+}(a)\} + \{\psi^{(k-1),-}(c)-\psi^{(k-1),+}(b)\}$$

$$= (D_k)\int_a^b f + (D_k)\int_b^c f$$

Theorem 3.2 *Let f be D_k integrable in [a,b] and let a<c<b. Let ϕ the k^{th} primitive of f in [a,b]. If $\phi_{ap}^{(k-1)}(c)$ exists then f is D_k integrable on [a,c] and on [c,b] and*

$$(D_k)\int_a^b f + (D_k)\int_b^c f = (D_k)\int_a^c f.$$

The proof is immediate.

Theorem 3.3 *Let f and g be D_k integrable in [a,b] and α, β are constants then $\alpha f + \beta g$ is D_k integrable in [a,b] and*

$$(D_k)\int_a^b (\alpha f + \beta g) = \alpha (D_k)\int_a^b f + \beta (D_k)\int_a^b g$$

The proof follows from the definition of the integral and Lemma 2.2.

Theorem 3.4 *If f be D_k integrable on [a,b], then f is measurable and finite a.e on [a,b].*

Proof: Since *f* is D_k integrable on [a,b], there is a continuous function $\phi \in AC_k G([a,b])$ such that $\phi_{ap}^{(r)}$ is smooth for $r = 1,2,...k-2$ and $\phi_{ap}^{(r)} = f, a.e$ on [a,b]. Let [a,b]= $\bigcup E_n$, where E_n is closed and ϕ is AC_k on E_n for each n. Let $E_n = P_n \cap D_n$ where P_n is perfect and D_n is countable. Since $\phi \in AC_k G(E_n)$, $\phi \in AC_k G(D_n)$ and so by Theorem-6(i) of [3], $\phi^{(k-1)}$ is AC_1 on P_n where the derivative $\phi^{(k-1)}$ is taken with respect to P_n . Let $\psi_n(x) = \phi^{(k-1)}(x)$ for $x \in P_n$ and ψ_n be linear in the closure of all intervals contiguous to P_n . Then $\psi_n \in BV_1[a,b]$ and so ψ_n' exists *a.e* on [a,b]. Also $f = \phi_{ap}^{(k)}(x) = \psi_n'(x)$ *a.e* in E_n . Since derivative of a function of bounded variation is finite *a.e* and measurable, *f* is finite *a.e* and is measurable on E_n for each *n*. The rest is clear.

Theorem 3.5 *If f is D_k integrable and f≥0 in [a,b], then f is Lebesgue integrable and the integrals are equal.*

Proof: Let ϕ be the k[th] primitive of *f*. So $\phi_{ap}^{(k)} = f \geq 0$ *a.e* in [a,b]. Then from Theorem-2.4, $\phi^{(k-1)}$ exists and is nondecreasing in [a,b] and $\phi^{(k-1)} \in AC_1([a,b])$. Hence $\phi^{(k)}$ exists *a.e* in [a,b] and is Lebesgue integrable in [a,b]. Hence *f* is Lebesgue integrable in [a,b].

Theorem 3.6 *(Integration by parts). Let $f:[a,b] \to R$ is D_k integrable in[a,b], F be its indefinite D_k integral and F is D^* integrable. Let $G^{(k-1)}$ be absolutely continuous in [a,b], then fG is D_k integrable in [a,b] and*

$$(D_k)\int_a^b fG = [FG]_a^b - (D^*)\int_a^b FG'$$

Proof: Let ϕ be the k[th] primitive of *f* in [a,b] .Let $\varphi(x) = \phi(x)G(x), x \in [a,b]$. Then since $\phi_{ap}^{(r)}$ is smooth for r=0,1,……,k-2 and $G^{(k-1)}$ exists finitely so $\psi_{ap}^{(r)}$ is smooth for r=0,1,…,k-2. Since $G^{(k-1)}$ is absolutely continuous ,G is AC_k in [a,b] and since ϕ is AC_kG in [a,b] . And a.e in [a,b]

$$\psi_{ap}^{(r)} = \phi G^{(k)} + {}^k c_1 \phi' G^{(k-1)} + ... + k\phi^{(k-1)} G' + \phi_{ap}^{(k)} G$$
$$= \phi G^{(k)} + {}^k c_1 \phi' G^{(k-1)} + ... + kFG' + fG$$

So $\phi G^{(k)} + {}^k c_1 \phi' G^{(k-1)} + ... + kFG' + fG$ *is* D_k integrable in [a,b].

Let $H(x) = \phi G^{(k-1)} + {}^{k-1}c_1 \phi' G^{(k-2)} + ... + (k-1)\phi^{(k-2)} G'$

Then in a similar way as that of Theorem-14 of [1] it can be proved that $H \in ACG^*([a,b])$.

Again since $\phi^{(k-1)}$ and $G^{(k)}$ exists a.e in [a,b],we have,

$$H'(x) = \phi G^{(k)} + ({}^{k-1}c_0 + {}^{k-1}c_1)\phi' G^{(k-1)} + ...(k-1)\phi^{(k-1)} G'$$
$$= \phi G^{(k)} + {}^k c_1 \phi' G^{(k-1)} + ... + (k-1)FG'$$

So $\phi G^{(k)} + {}^k c_1 \phi' G^{(k-1)} + ... + (k-1)FG'$ is D^* integrable in [a,b] and so D_k integrable and H is indefinite D_k integral. Since F is D^* integrable and G' is absolutely continuous in [a,b], FG' is D^* integrable by Theorem2.5(p.246) of [4] and hence FG' is D_k integrable in [a,b]. Again since, $(\phi G^{(k)} + {}^k c_1 \phi' G^{(k-1)} + ... + kFG' + fG)$, $(\phi G^{(k)} + {}^k c_1 \phi' G^{(k-1)} + ... + (k-1)FG')$ and FG' are D_k integrable, by Theorem3.3, *fG* is D_k integrable in [a,b] and

$$(D_k)\int_a^b fG = (D_k)\int_a^b (\phi G^{(k)} + {}^k c_1 \phi' G^{(k-1)} + \dots\dots + kFG' + fG)$$

$$-(D^*)\int_a^b (\phi G^{(k)} + {}^k c_1 \phi' G^{(k-1)} + \dots\dots + (k-1)FG') - (D^*)\int_a^b FG'$$

$$= [\psi_{ap}^{(k-1)}]_a^b - [H]_a^b - (D^*)\int_a^b FG'$$

$$= [\phi G^{(k-1)} + {}^{k-1} c_1 \phi' G^{(k-2)} + \dots\dots + (k-1)\phi^{(k-2)} G' + FG]_a^b$$

$$-[\phi G^{(k-1)} + {}^{k-1} c_1 \phi' G^{(k-2)} + \dots\dots + (k-1)\phi^{(k-2)} G']_a^b - (D^*)\int_a^b FG'$$

$$= [FG]_a^b - (D^*)\int_a^b FG'$$

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