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Bounds For The Mean Life Time Of A Lightly Loaded Standby System With Renewal

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ABSTRACT:- This paper deals with the average age estimation of the binary system, the bounds of the mean life time, the rate of failure parameter, some special cases and numerical examples are given. In the last section an estimation of the bounds of the reliability function is obtained

Keyword:- Mean Life Time, Reliability Function, Loaded Standby, Failure Rate, Lightly Loaded Standby

In this paper we derive an estimation for the lower and the upper bounds of the mean life time of the lightly loaded standby duplex system when the parameters of the repair time distribution function are known. Also a general inequality is obtained for the bounds of the mean life time an estimation of the rate of failure parameter is derived also. Some special cases and numerical examples are given. In the last section an estimation of the bounds of the reliability function is obtained.

I. RELIABILITY FUNCTIONS AND MEAN LIFE TIME OF THE LIGHTLY LOADED STANDBY DUPLEX SYSTEM.

In this section we investigate the reliability function and the mean life time of the lightly loaded standby duplex system with renewal, moreover some special cases are derived.

To increase the effectiveness of standby system, devices that have failed are repaired. Let us investigate the effect of repair on increasing the reliability. We confine ourselves to the case of one basic and one standby system.

To study the above system, let us assume that the following conditions are fulfilled:
(1) As soon as the main unit fails the standby unit assumes the load of it;
(2) The repair of the failed unit begins immediately after occurrence of failure;
(3) The repair fully restores the properties of the main unit that failed;
(4) The life time of the main unit is a random variable having exponential distribution with rate of failure $\lambda,(\lambda > 0)$. The life time of the standby unit is also a random variable having exponential law with rate of failure $\lambda, 0 \leq K \leq 1$;
(5) The repair time of the failed unit is a random variable with general distribution function f (.).
The cycle is defined as the length of time between two successive renewal. The failure of the system occurs when one unit fails while the other is still being repaired.

Let P (t) be the probability that the system works smoothly to the time t without failure. Let us also introduce the Laplace transforms

THEOREM 1: under the conditions (1) to (5), the function P(x) satisfies the following integral equation

$$P(t) = \exp\left(-\lambda \bar{k} t\right) + \lambda \bar{k} \int_0^t \exp\{-\lambda(kx+t)\} \bar{F}(t-x)\,dx + \lambda \bar{k} \int_0^t \int_0^x \exp\{-\lambda(kz+x)\}\, p(t-x)\,dF(x-z)\,dx \qquad (1.1)$$

In terms of Laplace transforms, the solution of this integral equation is given by the formula

$$p(s)=\frac{(\lambda+s)+\lambda\bar{k}\left[1-f(s+\lambda)\right]}{(\lambda+s)\left[s+\lambda\bar{k}\left(1-f(s+\lambda)\right)\right]} \qquad (1.2)$$

where $\bar{k}=1+k, 0\le k\le 1,$ and $\bar{F}(x)=1-F(x).$

PROOF: we prove this theorem by using the method in [1] As follows:

First we prove that P (t) satisfied the integral equation (1.1).The event that the pair will operate without failure during the interval (0, t) can be decomposed in to the following disjoint events:

(a) The first failure occurs after the instant t, the probability of' this event is $\exp\{-\lambda\bar{k}t\}$.

(b) The first failure occurs prior to the instant t but the first cycle ends after the instant t. The standby unit that is put into operation at the instant of failure operates without failure up to the instant t. the probability of this event is

$$\lambda\bar{k}\int_0^t \exp\{-\lambda(kx+t)\}\bar{F}(t-x)dx.$$

(c) The first cycle ends prior to the instant t. the second unit operates without failure during the first renewal period, and the pair operates without failure for the remainder of the time until the instant t, the probability of this event is

$$\lambda\bar{k}\int_0^t\int_0^x \exp\{-\lambda(kz+x)\}\, p(t-x)dF(x-z)dz.$$

Adding these three probabilities, we finally obtain the desired integral equation (1.1).
By taking the Laplace transforms of equation (1.1), we easily obtain equation (1.2), and hence the proof of the theorem is completed.

If Z is the random life time of a pair then $p(t)=p\{Z>t\}.$ from (1.2) it is easy to see that the mean life time of the system is as follows:

$$T=\int p(t)dt=p(0)=\frac{1+\bar{k}\left[1-f(\lambda)\right]}{\lambda\bar{k}\left[1-f(\lambda)\right]}=\frac{1}{\lambda}+\frac{1}{\lambda\bar{k}a} \qquad (1.3)$$

where ; $a=\int_0^\infty\left(1-e^{-\lambda t}\right)dF(t).$hence applying (1.2) we have the variance of the life time of the system

$$\sigma^2 = 2\int_0^\infty tP(t)dt - T^2 = \frac{1}{\lambda^2} + \frac{1-2\lambda\bar{k}f'(\lambda)}{\lambda^2\bar{k}^2[1-f(\lambda)]^2} \geq \frac{1}{2}T^2 \tag{1.4}$$

In particular, for anon loaded and for a loaded as special cases of (1.3) and (1.4), we have respectively

i. If K=0, then

$$T = \frac{1}{\lambda} + \frac{1}{\lambda a} \quad and \quad \sigma^2 = \frac{1}{\lambda^2} + \frac{1-2\lambda f'(\lambda)}{\lambda^2 a^2}$$

which are respectively the mean and variance for the Life time of a duplex system with renewal in the case of a non-loaded standby system (say the cold standby system).

ii. If K =1. Then

$$T = \frac{1}{\lambda} + \frac{1}{2\lambda a} \quad \text{and} \quad \sigma^2 = \frac{1}{\lambda^2} + \frac{1-4\lambda F'(\lambda)}{4\lambda^2 a^2}.$$

which are respectively the mean and the variance for the life time of a loaded standby duplex system with renewal? (Say, the hot standby system).
If the System is without renewal then the mean and the variance for the life time of a lightly loaded standby duplex system are respectively

$$T_0 = \frac{1}{\lambda} + \frac{1}{\lambda\bar{k}}, \quad \sigma_0^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda^2\bar{k}^2} \tag{1.5}$$

Taking k=0 in (1.5) we get $T_0 = \frac{2}{\lambda}$ (the mean life time of a cold standby duplex system without renewal) and $\sigma_0^2 = \frac{2}{\lambda^2}$. taking k=1 in (1.5) we get $T_0 = \frac{3}{2\lambda}$ (the mean life time of a hot standby duplex system without renewal) and $\sigma_0^2 = \frac{5}{4\lambda^2}$.

The gain in mean life time provided by the renewal in lightly loaded standby duplex system is

$$\frac{T}{T_0} = \frac{\bar{k}a+1}{(1+\bar{k}a)}. \tag{1.6}$$

The smaller the probability "a" of failure of a pair in a single cycle, the greater would be the gain.

II. ESTIMATION OF THE LOWER AND UPPER BOUNDS FOR THE MEAN LIFE TIME OF A LIGHTLY LOADED STANDBY DUPLEX SYSTEM.

This section contains the fundamental results in this paper.
In the proceeding section it has been shown that the mean life time of the system is given by

$$T = \frac{1}{\lambda} + 1 \Big/ \left\{ \lambda (1+k) \int_0^\infty \left(1 - e^{-\lambda t}\right) df\,(t) \right\}. \tag{2.1}$$

Applying (2.1), we show that the values of T depend on the parameters of the distribution F(t), and we derive the lower and upper bounds of T in the following different cases.

1. The case when no assumptions are considered on the distribution function F (T).

In this case

$$\max_F \int_0^\infty e^{-\lambda t} dF(t) = 1,$$

$$\min_F \int_0^\infty e^{-\lambda t} dF(t) = 0.$$

The first relation is satisfied if we write F(t) = e(t) where e(t) =0 if $t < 0$ and e(t) = 1 if $t \geq 0$. The second relation. Applies if the function F (t) takes the form

$$F_n(t) = e(t-n) = \begin{cases} 0, & if \quad t<n, \\ 1, & if \quad t \geq n, \end{cases}$$

Since $$\int_0^\infty e^{-\lambda t} dF_n(t) = e^{-\lambda n} \to 0 \ \ as \ \ n \to \infty$$

Then from (2.1) we obtain, In this case $\frac{1}{\lambda}\frac{1+\bar{k}}{\bar{k}} < T < \infty$.

2- The case when the mathematical expectation of the distribution function F (t) is known.

We denote the mathematical expectation of the function F(t) by

$$\frac{1}{\mu} = \int_0^\infty t dF(t).$$

Let the class of the distribution function that satisfy this condition be K_1. In this case we have

$$\sup_{F \in k_1} \int_0^\infty e^{-\lambda t} df(t) = 1 \quad \text{and} \quad \inf_{F \in k_1} \int_0^\infty e^{-\lambda t} dF(t) = e^{-\rho}$$

where ; $\rho = \frac{\lambda}{\mu}$.

To prove these two relations, we take for the first relation the sequence $F_n(t)$ in the form

$$F_n(t) = \left(1 - \frac{1}{n}\right) e(t) + \frac{1}{n} e\left(t - \frac{n}{\mu}\right)$$

where ; e(t) is defined as above (case 1). It is easy to check that $F_n \in k_1$. Thus

$$\int_0^\infty e^{-\lambda t} dF_n(t) = \left(1 - \frac{1}{n}\right) + \frac{1}{n} e^{-n\rho} \to 1 \; as \;\; n \to \infty.$$

The proof of the second relation follows directly from the known inequality

$$\int_0^\infty e^{-\lambda t} dF(t) \geq \exp\left\{-\lambda \int_0^\infty t dF(t)\right\} e^{-\rho};$$

Since $e^{-\lambda t}$ is a convex function from below the equality

$$\int_0^\infty e^{-\lambda t} dF(t) = e^{-\rho} \tag{2.2}$$

is satisfied. For $F(t) = e\left(t - \frac{1}{\mu}\right)$. Thus from (2.1) we obtain, in this case,

$$\frac{1}{\lambda}\left[1 + \frac{1}{\bar{k}\left(1 - e^{-\rho}\right)}\right] \leq T < \infty.$$

III. THE CASE WHEN THE MEAN AND THE VARIANCE OF THE DISTRIBUTION FUNCTION ON F (T) ARE KNOWN

We denote the mathematical expectation and the variance of

the distribution function F (t) by

$$\frac{1}{\mu} = \int_0^\infty t dF(t),$$
$$\sigma^2 = \int_0^\infty \left(t - \frac{1}{\mu}\right)^2 dF(t) > 0 \tag{2.3}$$

And so

$$\int_0^\infty t^2 dF(t) = \left(\frac{1}{\mu}\right)^{2+\sigma^2 = m_2}, \text{ say .}$$

Let the class of the distribution functions F (t) that satisfy (2.3) be denoted by K_2. In this case we have the following theorem.

THEOREM 2:

$$\inf_{F\in K_2}\int_0^\infty e^{-\lambda t}dF(t)=e^{-\rho} \tag{2.4}$$

$$\sup_{f\in k_2}\int_0^\infty e^{-\lambda t}dF(t)=\max_{F\in k_2}\int_0^\infty e^{-\lambda t}dF(t)=$$

$$=\frac{\sigma^2}{m_2}+\frac{e^{-\lambda m_2\mu}}{m_2\mu^2}\frac{1}{b}\left[\mu^2\sigma^2+e^{-\rho b}\right] \tag{2.5}$$

where ; $b=1+\mu^2\sigma^2.$

PROOF: To prove the first equality we note that the relation (2.2) is tru*e*. And we take the sequence of the distribution. Functions $F_n(t)\in k_2$ in the form

$$F_n(t)=p_1^{(n)}e\left(t-t_1^{(n)}\right)+p_2^{(n)}e\left(t-t_2^{(n)}\right);$$

Then from (2.3) it follows that

$$p_1^{(n)}+p_2^{(n)}=1,$$
$$p_1^{(n)}t_1^{(n)}+p_2^{(n)}t_2^{(n)}=\frac{1}{\mu}, \tag{2.6}$$
$$p_1^{(n)}t_1^{(n)^2}+p_2^{(n)}t_2^{(n)^2}=m_2.$$

Suppose that $t_1^{(n)}=\frac{1}{\mu}-\frac{1}{n},$ then from (2.6) we obtain

$$t_2^{(n)}=\frac{1}{\mu}+n\sigma^2,$$

$$p_1^{(n)}=\frac{n^2\sigma^2}{1+n^2\sigma^2},$$

$$p_2^{(n)}=\frac{1}{1+n^2\sigma^2},$$

from which

$$\int_0^\infty e^{-\lambda t} dF(t) = p_1^{(n)} e^{-\lambda t(n)} + p_2^{(n)} e^{-\lambda t_2^{(n)}} =$$

$$\frac{n^2\sigma^2}{1+n^2\sigma^2} e^{-\lambda\left(\frac{1}{n}+n\sigma^2\right)} \to e^{-\rho}, as\, n \to \infty.$$

To prove the second equality we introduce two subclasses M_n, N_n of the class K_2 such that

$$F(t) = \sum_{i=1}^{n} p_i\, e(t - t_i), if\ \ F(t) \in M_n$$

and

$$F(t) = \sum_{i=1}^{n-1} p_i\, e(t - t_i) + p_n e(t), if\ \ F(t) \in N_n;$$

Thus F (t) satisfies (2.2) for each subclass. Now we prove the following two lemmas.

LEMMA 1:

$$\sup_{F \in M_2} \int_0^\infty e^{-\lambda t} dF(t) = \max_{F \in M_2} \int_0^\infty e^{-\lambda t} dF(t) =$$

$$= \max_{F \in N_2} \int_0^\infty e^{-\lambda t} dF(t) = \frac{\sigma^2}{m_2} + \frac{1}{m_2\mu^2} e^{-\lambda m_2 \mu} \qquad (2.7)$$

PROOF: If $F(t) \in M_2$, then

$$\int_0^\infty e^{-\lambda t} dF(t) = p_i\, e^{-\lambda t_1} + p_2 e^{-\lambda t_2}$$

and

$$p_1 + p_2 = 1,$$
$$p_1 t_1 + p_2 t = \frac{1}{\mu}, \qquad (2.8)$$
$$p_1 t_1^2 + p_2 t_2^2 = m_2, p_1, p_2, t_1, t_2 \ge 0.$$

From the condition $\sigma^2 > 0$ it follow that $p_1 \ne 0$ and $t_1 \ne t_2$. Then from (2.8) we obtain

$$t_1 = \frac{1}{\mu} + \frac{\sigma^2}{\frac{1}{\mu} - t_2},$$

$$p_1 = \frac{\left(\frac{1}{\mu} - t_2\right)^2}{\sigma^2 + \left(\frac{1}{\mu} - t_2\right)^2},$$

$$p_2 = \frac{\sigma^2}{\sigma^2 + \left(\frac{1}{\mu} - t_2\right)^2}.$$

since $t_1 \geq 0$, then either $t_2 < \frac{1}{\mu}$ or $t_2 \geq m_2\mu.$ owing to the symmetry of the values of $t_1\,and\,t_2$, it is sufficient to investigate the case when $0 \leq t_2 < \frac{1}{\mu},$ then

$t_1 \geq m_2\mu > t_2$

and

$$\int_0^\infty e^{-\lambda t} dF(t) = \frac{\left(\frac{1}{\mu} - t_2\right)^2}{\sigma^2 + \left(\frac{1}{\mu} - t_2\right)^2} e^{-\lambda\left(\frac{1}{\mu} + \frac{\sigma 2}{\frac{1}{\mu} - t 2}\right)} +$$

$$\frac{\sigma^2}{\sigma^2 + \left(\frac{1}{\mu} - t_2\right)^2} e^{-\lambda t 2} = G(t_2), \quad say\ .$$

We show that $G\,(t_2)$ is a decreasing function in t_2 in the interval $0 \leq t_2 < \frac{1}{\mu}$. In fact, since

$$G'(t_2) = \frac{\sigma^2 e^{-\lambda\, t_2}}{\sigma^2 + \left(\frac{1}{\mu} - t_2\right)^2} \left[-\frac{2}{t_1 - t_2} e^{-\lambda(t_1 - t_2)} - \lambda e^{-\lambda(t_1 - t_2)} + \frac{2}{t_1 - t_2} - \lambda \right] =$$

$$= \frac{\sigma^2 e^{-\lambda\ t_2}}{\sigma^2 + \left(\frac{1}{\mu} - t_2\right)} g(\lambda), g(0) = 0.$$

Then

$$g'(\lambda) = -1 + e^{-\lambda\ (t_1 - t_2)} + \lambda(t_1 - t_2) e^{-\lambda\ (t_1 - t_2)} < 0 \quad ,$$

if $\lambda > 0$ and $t_1 - t_2 > 0$. consequently $g(\lambda) < 0$ and $G'(t_2) < 0,$

i.e. , $G(t_2)$ is a decreasing function attaining its maximum at $t_2 = 0$.the proof of the lemma then directly.If we modify the relation in (2.8) to:

$$\begin{aligned} p_1 + p_2 &= p^* \\ p_1 t_1 + p_2 t_2 &= m_1^*, \\ p_1 t_1^2 + p_2 t_2^2 &= m_2^*, \end{aligned} \tag{2.9}$$

$$p_1, p_2, t_1, t_2 \geq 0,\ 0 \leq p^* \leq 1,\ m_2^* \geq \left(m_1^*\right)^2,$$

then the proceeding lemma leads to the following lemma,

LEMMA 2:

$$\max\left(p_1 e^{-\lambda\ t_1} + p_2 e^{-\lambda\ t_2}\right) = \max\left(p_1 e^{-\lambda\ t_1} + p_2\right).$$

PROOF: the maximum on the left hand side is obtained by taking; $\{p_1, p_2, t_1, t_2\}$ to satisfy (2.9). But the maximum on the right hand side is obtained by taking $\{p_1, p_2, t_1, t_2 = 0\}$, also to satisfy (2.9). The proof of the lemma is obvious when $p^* = 0.$ If $p^* \neq 0$, then from (2.9) we get p_1, p_2 in terms of p^* (this reduces to the system (2.8) of the first lemma). For $m_2^* p^* = \left(m_1^*\right)^2$, the proof of the lemma is also trivial. For $m_2^* p^* \geq \left(m_1^*\right)^2$ we arrive to the conditions of the first lemma. Thus the proof of the second lemma is complete. Now we are going to prove the second equality (2.5) in theorem 2. We have for all $F \in M_n$

$$F(t)=\sum_{i=1}^{n} p_i\, e(t-t_i),$$
$$\sum_{i=1}^{n} p_i = 1$$
$$\sum_{i=1}^{n} p_i t_i = \frac{1}{\mu} \qquad (2.10)$$
$$\sum_{i=1}^{n} p_i t_i^2 = m_2,$$

and

$$\max_{F\in M_n}\int_0^\infty e^{-\lambda t}\,dF(t)=\max\sum_{i=1}^{n} p_i e^{-\lambda t_i},$$

where the maximum on the right hand side is obtained by taking $\{p_1, p_2, \ldots, p_n, t_1, t_2, \ldots, t_n\}$ to satisfy (2.9). Let the class $\{p_1, p_2, \ldots, p_{n-2}, t_1, t_2, \ldots, t_{n-2}\}$ which satisfies (2.9) with the absence of $p_{n-1}, p_n, t_{n-1}, t_n$ be denoted by c_{n-2}, then

$$\max_{F\in M_n}\int_0^\infty e^{-\lambda t}\,dF(t)=$$
$$\max\left\{\max\int_0^\infty e^{-\lambda t}\,dF(t)\right\}\left\{\begin{matrix} p_1,\ldots\ldots, p_{n-2} \\ t_1,\ldots\ldots\ldots, t_{n-2}\end{matrix}\right\}\left\{\begin{matrix} p_{n-1}, p_n \\ t_{n-1}, t_n\end{matrix}\right\}$$

Where $p_{n-1}, p_n, t_{n-1}, t_n$ satisfy the following relations

$$p_{n-1}+p_n = 1-\sum_{i=1}^{n-2} p_i = p^*,$$
$$p_{n-1}t_{n-1}+p_n t_n = \frac{1}{\mu}-\sum_{i-1}^{n-2} p_i t_i = m_1^*,$$
$$p_{n-1}t_{n-1}^2+p_n t_n^2 = m_2-\sum_{i-1}^{n-2} p_i t_i^2 = m_2^* \qquad (2.11)$$

with some fixed $p_1, \ldots, p_{n-2}, t_1, t_2, \ldots, t_{n-2}$.

LEMMA 2

$$\max \int_0^\infty e^{-\lambda t} dF(t) \begin{Bmatrix} p_{n-1}, & p_n \\ t_{n-1}, & t_n \end{Bmatrix}$$

Is obtained at $t_n = 0,$ from which it follow that

$$\max_{F \in M_n} \int_0^\infty e^{-\lambda t} dF(t) = \max_{F \in M_n} \int_0^\infty e^{-\lambda t} dF(t).$$

Using lemma 1 and 2 successively we obtain

$$\max_{F \in N_n} \int_0^\infty e^{-\lambda t} dF(t) = \max_{F \in N_{n-1}} \int_0^\infty e^{-\lambda t} dF(t) = ... =$$

$$= \max_{F \in N_2} \int_0^\infty e^{-\lambda t} dF(t) = \frac{\sigma^2}{m_2} + \frac{1}{m_2 \mu^2} e^{-\lambda m_2 \mu}$$

If $F(t)$ is arbitrary distribution function which satisfies (2.2), then by the definition of Stieltjes integral we have

$$\int_0^\infty e^{-\lambda t} dF(t) = \lim_{n \to \infty} \left\{ \sum_i p_1^{(n)} e^{-\lambda t_i^n} \right\} \leq$$

$$\leq \lim_{n \to \infty} \left\{ \frac{\sigma_n^2}{m_2^{(n)}} + \frac{1}{\mu_n^2 m_2^{(n)}} e^{-\lambda m_2^{(n)} \mu_n} \right\} =$$

$$= \frac{\sigma^2}{m_2} + \frac{1}{m_2 \mu} e^{-\lambda m_2 \mu}.$$

Thus we have proved that

$$\max_{F \in k_2} \int_0^\infty e^{-\lambda t} dF(t) = \frac{\sigma^2}{m_2} + \frac{1}{m_2 \mu^2} \exp\{-\rho b\}$$

$$= \frac{1}{b}\left[\mu^2\sigma^2 + \exp\{-\rho b\}\right].$$

Then it follows from (2.1) that

$$\frac{1}{\lambda}\left[1 + \frac{1}{\bar{k}\left(1 - e^{-\rho}\right)}\right] < T \le \frac{1}{\lambda}\left[1 + \frac{b}{\bar{k}\left(1 - e^{-\rho b}\right)}\right] \qquad (2.12)$$

Some results obtained as special cases of inequalities (2.12) are as follows:

(i) If the system is without renewal then the lower bound is equal to the upper bound of the mean life time T and equal to $\frac{1+\bar{k}}{\bar{k}}\frac{1}{\lambda}$.

(ii) If the repair time distribution is constant, i.e., if F(t) =1 for t ≥R , F(t) = 0 for t< R, then the mean repair time $\frac{1}{\mu} = R$ and the variance $\sigma^2 = 0;$ substituting in (2.12) we get the lower bound of the mean life time equal to its upper bound and is equal to

$$T = \frac{1}{\lambda} + \left[1 + \frac{1}{\bar{k}\left(1 - e^{-\lambda R}\right)}\right]$$

(iii) If $F(t) = 1 - e^{-t/R}$, t > 0 , then $\frac{1}{\mu} = R$ *and* $\sigma^2 = R^2$ and substituting in (2.12) we get

$$\frac{1}{\lambda}\left[1 + \frac{1}{\bar{k}\left(1 - e^{-\theta}\right)}\right] < T \le \frac{1}{\lambda}\left[1 + \frac{2}{\bar{k}\left(1 - e^{-2\theta}\right)}\right].$$

Where $\theta = \lambda R > 0.$ in this case we have worked out two numerical examples. These examples are given in the following tables.

$$\lambda = 1,\ k = 1/4$$

θ	L.B of T	U.B of T
0.02	68.3360	69.0094
0.04	35.0047	35.6845
0.06	23.8957	24.5823
0.08	18.3423	19.0356
2	2.5420	3.7164
4	2.3582	3.6847
6	2.3366	3.6668
8	2.3338	3.6667

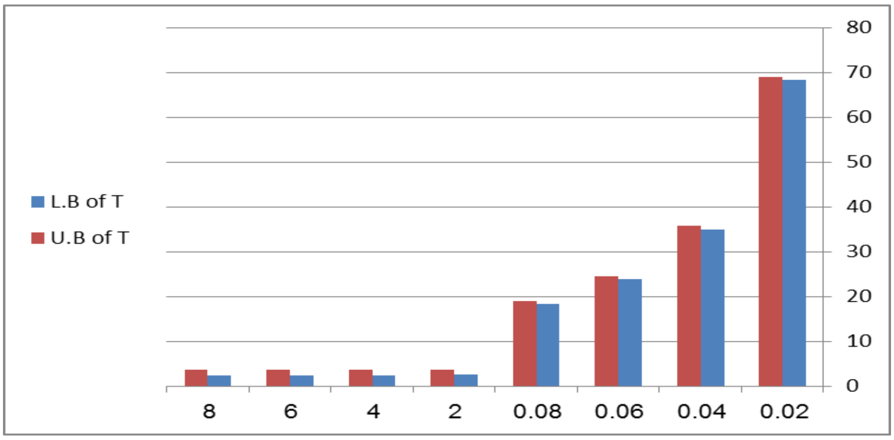

Figure 1

$$\lambda = 2 \, , \, k = \tfrac{1}{2}$$

θ	L.B of T	U.B of T
0.02	51.0020	51.507
0.04	26.0036	26.5134
0.06	17.6717	18.1867
0.08	13.5067	14.0267
2	1.6565	2.5373
4	1.5187	2.5007
6	1.5002	2.5000
8	1.5003	2.5000

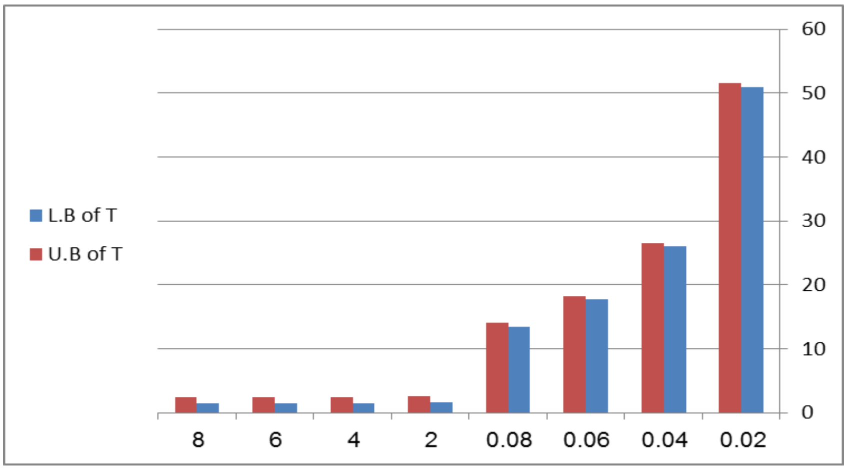

Figure 2

(iv) if F(t) has a gamma distribution, i.e., if

$$F(t)=\frac{t}{R\Gamma(n)}\left(\frac{t}{R}\right)^{n-1}e^{-t/R},\ t>0,\ R>0,$$

Then $\frac{1}{\mu}=nR$ and $\sigma^2=nR^2$ and substituting in to (2.12) we get

$$\frac{1}{\lambda}\left[1+\frac{1}{\bar{K}\left(1-e^{-n\,\theta}\right)}\right]<T\le\frac{1}{\lambda}\left[1+\frac{n+1}{n\bar{k}\left(1-e^{-(n+1)\,\theta}\right)}\right]$$

(2.13)

where ; $\theta=\lambda R.$ in this case we have worked out the numerical example given in the following table:

$$\lambda=1\ ,\ K=\frac{1}{4}\ ,\ n=2$$

θ	**L.B of T**	**U.B of T**
0.02	35.0047	**35.3435**
0.04	18.3423	**28.6867**
0.06	12.7911	**13.1411**
2	2.35820	**3.0050**
4	2.3338	**3.0000**
6	2.3333	**3.0000**

(4) The case when the first n moments of the distribution function F(T) are known. let them be $m_1,m_2,.....,m_n$ where

$$m_k=\int_0^\infty t^k dF(t).$$

If the distribution function F(t) be with non-decreasing hazard rate in t, $\left[F(t+x)-F(t)\right]/\left[1-F(t)\right]$ for any fixed x > 0, then the following relations are satisfied

$$1\ge\frac{m_2}{2!m_1^2}\ge\frac{m_3}{3!m_1^3}\ge....\ge\frac{m_n}{n!m_1^n}\ge.... \qquad (2.14)$$

If the first three moment of the distribution function F (t) are known, then from (1.3) and (2.14) we obtain, in this case the following inequality

$$\left[\lambda m_1\left\{1-\frac{\lambda m_2}{2!m_1}-\frac{(4-\lambda)\lambda^2}{4!}\frac{m_3}{m_1}\right\}\right]^{-1} < T' < \left[\frac{m_1^2}{m_2}\left(1-e^{-\frac{\lambda m_2}{m_1}}\right)\right]^{-1}$$

where $T' = \bar{k}\lambda\left(T - \dfrac{1}{\lambda}\right).$ (2.15)

Generally let the first moment $m_1, m_2, \ldots\ldots, m_n$ of the function F(t) be given and satisfy (2.14). Then using the result obtained in (2.15) it is not difficult to obtain exactly the estimation for the values of T'. for making $\lambda \to 0$ in (2.15) we get the following results:

$$1 < T' < \infty, \qquad (n=0)$$

$$\left(1-\frac{\lambda m_2}{2!m_1}\right)^{-1} < \lambda m_1 T' < \left(1-\frac{\lambda \bar{m}_2}{2!m_1}\right)^{-1} \qquad (n=1)$$

$$\left(1-\frac{\lambda m_2}{2!m_1}+\frac{\lambda^2 \bar{m}_3}{3!m_1}\right)^{-1} < \lambda m_1 T' < \left(1-\frac{\lambda m_2}{2!m_1}+\frac{\lambda^2 m_3}{3!m_1}\right)^{-1} \qquad (n=2)$$

$$\left(1-\frac{\lambda m_2}{2!m_1}+\frac{\lambda^2 m_3}{3!m_1}-\frac{\lambda^3 \bar{m}_4}{4!m_1}\right)^{-1} < \lambda m_1 T' < \left(1-\frac{\lambda m_2}{2!m_1}+\frac{\lambda^2 m_3}{3!m_1}-\frac{\lambda^2 \bar{m}_4}{4!m_1}\right)^{-1} \qquad (n=3)$$

Now we prove these inequalities. The first inequality is obvious. For the others we have

$$a = \lambda m_1\left[1-\frac{\lambda m_2}{2!m_1}+\frac{\lambda^2 m_3}{3!m_1}-\frac{\lambda^3 m_4}{4!m_1}+\ldots\ldots\right]$$

and $\lambda m_1 T' = \dfrac{1}{a'},$ where

$$a' = 1-\frac{\lambda m_2}{2!m_1}+\frac{\lambda^2 m_3}{3!m_1}-\frac{\lambda^3 m_4}{4!m_1}+\ldots\ldots \quad .$$

For n=1, we have

$$a' = 1-\frac{\lambda m_2}{2!m_1} \quad .$$

Since $m_1^r \le m_r$ then

$$a' < 1-\frac{\lambda m_1}{2!}, i.e. \frac{1}{a'} > \left(1-\frac{\lambda m_1}{2!}\right)^{-1} = \left(1-\frac{\lambda m_2}{2! m_1}\right)^{-1}$$

where , $m_2 = m_1^2$, hence,

$$\lambda m_1 T' > \left(1-\frac{\lambda m_2}{2! m_1}\right)^{-1}.$$

Also since $1 \geq \frac{m_2}{2! m_1^2}$, then,

$$a' > 1-\lambda m_1, \frac{1}{a'} < (1-\lambda m_1) = \left(1-\frac{\lambda \overline{m}_2}{2! m_1}\right)^{-1};$$

$\overline{m}_2 = 2m_1^2$, So that

$$\lambda m_1 T' < \left(1-\frac{\lambda \overline{m}_2}{2! m_1}\right)^{-1}.$$

Therefore

For n=1 , $\left(1-\frac{\lambda m_2}{2! m_1}\right)^{-1} < \lambda m_1 T' < \left(1-\frac{\lambda \overline{m}_2}{2! m_1}\right)^{-1}$.

For n=2 , $a' = 1-1-\frac{\lambda m_2}{2! m_1} + 1-\frac{\lambda^3 m_3}{3! m_1},$

since $\frac{\lambda m_3}{3! m_1^3} < \frac{m_2}{2! m_1^2}$ *then* $\frac{m_3}{3! m_1} < \frac{m_2}{2}$

then $$a' < \left(1-\frac{\lambda m_2}{2! m_1} + \frac{\lambda^2 m_2}{2}\right)$$

and $$\frac{1}{a'} > \left(1-\frac{\lambda m_2}{2! m_1} + \frac{\lambda^2 \overline{m}_3}{3! m_1}\right)^{-1}.$$

Hence

$$\lambda m_1 T' > \left(1-\frac{\lambda m_2}{2! m_1} + \frac{\lambda^2 \overline{m}_3}{3! m_1}\right)^{-1}, \overline{m}_3 = 3 m_2 m_1 :$$

also $$a \geq \lambda m_1 \left(1 - \frac{\lambda m_2}{2!m_1} + \frac{\lambda^2 m_2^2}{3!m_1^2}\right)^{-1}$$

and so $$a \geq \lambda m_1 \left(1 - \frac{\lambda m_2}{2!m_1} + \frac{\lambda^2 m_3}{3!m_1}\right)^{-1}, m_3 = \frac{m_2^2}{m_1}$$

Then

$$\lambda m_1 T' < \left(1 - \frac{\lambda m_2}{2!m_1} + \frac{\lambda^2 m_3}{3!m_1}\right)^{-1} .$$

Therefore

$$\left(1 - \frac{\lambda m_2}{2!m_1} + \frac{\lambda^2 \overline{m}_3}{3!m_1}\right)^{-1} < \lambda m_1 T' < \left(1 - \frac{\lambda m_2}{2!m_1} + \frac{\lambda^2 m_3}{3!m_1}\right)^{-1} \quad (n = 2)$$

The proof of the last inequality is similar to the proof of the previous inequalities, and in general it can be proved that

$$\frac{1}{\underline{\alpha}(n)} < \lambda m_1 T' < \frac{1}{\overline{\alpha}(n)}, \quad \text{when n is odd}$$

where

$$\underline{\alpha}(n) = \sum_{r=1}^{n} (-)^{r-1} \frac{\lambda^{r-1}}{r!} \frac{m_r}{m_1} - \frac{\lambda^n \underline{m}_{n+1}}{(n+1)!m_1},$$

$$\overline{\alpha}(n) = \sum_{r=1}^{n} (-)^{r-1} \frac{\lambda^{r-1}}{r!} \frac{m_r}{m_1} - \frac{\lambda^n \overline{m}_{n+1}}{(n+1)!m_1} .$$

Also

$$\frac{1}{\underline{B}(n)} < \lambda m_1 T' < \frac{1}{\overline{B}(n)} \quad \text{n. even and } \geq 2 ,$$

where;

$$\underline{B}(n) = \sum_{r=1}^{n} (-)^{r-1} \frac{\lambda^{r-1}}{r!} \frac{m_r}{m_1} - \frac{\lambda^n \overline{m}_{n+1}}{(n+1)!m_1},$$

$$\overline{B}(n) = \sum_{r=1}^{n} (-)^{r-1} \frac{\lambda^{r-1}}{r!} \frac{m_r}{m_1} - \frac{\lambda^n m_{n+1}}{(n+1)!m_1} .$$

Now, let

$$D_n = T_{\max}^{(n)} - T_{\min}^{(n)} = \frac{T_{\max}^{\prime(n)} - T_{\min}^{\prime(n)}}{\overline{k}\lambda}$$

be the difference between the upper $T_{\max}^{(n)}$ and the lower $T_{\min}^{(n)}$ bounds of the mean life time T(n) , when the first n initial moments of the repair distribution function are given, then

$$D_0 = \infty$$

$$D_1 = \frac{1}{\overline{k}\lambda} \frac{\overline{m}_2 - m_2}{2!m_1^2},$$

$$D_2 = \frac{1}{\overline{k}} \frac{\overline{m}_3 - m_3}{3!m_1^2},$$

$$D_3 = \frac{\lambda}{\overline{k}} \frac{\overline{m}_4 - m_4}{4!m_1^2},$$

and in general

$$D_n = \frac{\lambda^{n-2}}{\overline{k}} \frac{\overline{m}_{n+1} - m_{n+1}}{n+1!m_1^2}, n \geq 1$$

where

$$\overline{m}_n = n m_{n-1} m_1, \ \underline{m}_1 = 0, \ \underline{m}_2 = m_1^2, \underline{m}_3 = \frac{m_2^2}{m_1},$$

$$\underline{m}_4 = m_3, \ldots \qquad .$$

Thus for the distribution function F(t) it follows that when $\lambda \to 0$ the difference between the estimations of the upper and the lower bounds for the mean life time of the lightly loaded standby duplex system is equal to ∞ for any n=0 , tends to ∞ for any n=1, tends to a definite non-zero value for n=2 and tends to zero with n≥3 .

THEOREM 3: If the distribution function F(t) of the repair time for the failed unit satisfies (2.14) , then we can estimate the rate of failure λ as follows :

$$\hat{\lambda}(n) = \frac{m_{2n} - 2n.m_{(2n-1)}.m_1}{m_1 m_{2n}}, n = 1,2,3,\ldots.$$

PROOF: From (1.3) and (2.14), it is not difficult to obtain the following inequality:

$$\frac{1}{\underline{s}n} < \lambda m_1 T' < \frac{1}{\overline{s}n} \qquad (2.16)$$

Where

$$\bar{s}(n)=\sum_{r=1}^{2n-1}(-)^{r-1}\lambda^{r-1}\frac{m_r}{r!m_1}-\frac{\lambda^{2n-1}\overline{m}_{2n}}{(2n)!m_1}$$

and

$$\underline{s}(n)=\sum_{r=1}^{2n}(-)^{r-1}\lambda^{r-1}\frac{m_r}{r!m_1}+\lambda^{2n}\frac{\overline{m}_{(2n+1)}}{(2n+1)!m_1}$$

where $$\overline{m}_r=r\,m_{r-1}m_1,r=1,2,3,.....\ .$$

If $\bar{s}(n)=\underline{s}(n)$ we have

$$\hat{\lambda}(n)=(2n+1)\frac{m_{2n}-\overline{m}_{2n}}{\overline{m}_{(2n+1)}}=\frac{m_{2n}-(2n)m_{(2n-1)}m_1}{m_1m_{2n}},n\geq 1$$

This is an estimation of the rate of failure λ. For example

$$\hat{\lambda}(1)=\frac{m_2-2m_1^2}{m_1m_2}\qquad (n=1)$$

$$\hat{\lambda}(2)=\frac{m_4-4m_1-m_3}{m_1m_4}\qquad (n=2)$$

and so on .

IV. LIMITING DISTRIBUTION FOR THE RELIABILITY FUNCTION OF THE STANDBY DUPLEX SYSTEM.

In this section we show that the reliability function P(t) in (1.1) tends to the exponential distribution, moreover we estimate this function in the limiting case;

THEOREM 4 : If the lightly loaded standby system satisfy the conditions (1) to (5) and moreover

$$a=\int_0^{\infty}\left(1-e^{-\lambda t}\right)dF(t)\rightarrow 0,$$

then

$$p\{a\tau>t\}\rightarrow\exp\left(-\lambda\bar{k}t\right)\quad , \tag{3.1}$$

where τ is the interval of time between two successive failures.

We can prove this theorem by using the method in [2],from (3.1) it follows that, the limiting distribution of the reliability function for small "a" is approximately

$$p(t) \approx e^{-\lambda \bar{k} a t} \tag{3.2}$$

further investigation shows that it is better to use the approximation formula

$$p(t) \approx e^{-\frac{t}{T}} \tag{3.3}$$

We investigation the estimation for p(t) in the limiting case.
We have for small (t/T) the following approximation

$$\hat{p}(t) \approx 1 - \frac{t}{T} + 0\left(\frac{t^2}{T^2}\right) \tag{3.4}$$

Then we have from (2.3) the following inequality

$$1 - \frac{t}{T_{lower}} < \hat{p}(t) < 1 - \frac{t}{T_{upper}} \tag{3.5}$$

We further show that the values of $\hat{p}(t)$ depend on the parameters of the distribution function for the repair time of the failed unit ,and we derive the bounds of $\hat{p}(t)$ in the following different cases:

1) The case when no assumptions, are considered on the distribution function of repair time of the failed unit F(t) . from case 1 in section (3.2) and from (3.5) , we have

$$1 - \frac{\bar{k}\lambda t}{\left(1+\bar{k}\right)} < \hat{p}(t) < 1.$$

2) The case when the mathematical expectation of the distribution function F(t) is known. From case 2 in section (3.2) and from (3.5) . we have

$$1 - \frac{\bar{k}\lambda t\left(1-e^{-\rho}\right)}{1+\bar{k}\left(1-e^{-\rho}\right)} < \hat{p}(t) < 1.$$

3) The case when the mean and the variance of the distribution function F(t) are known. From (2.12) and (3.5) we have

$$1 - \frac{\bar{k}\lambda t\left(1-e^{-\rho}\right)}{1+\bar{k}\left(1-e^{-\rho}\right)} < \hat{p}(t) < 1 - \frac{\bar{k}\lambda t\left(1-e^{-\rho b}\right)}{\bar{k}\left(1-e^{-\rho b}\right)+b}$$

Lastly we give the following result:
If the distribution function F(t) satisfies the relation (2.14) , then we have the following inequality

$$\frac{1+(1-t\lambda)\underline{B}(n)}{1+\underline{B}(n)} < \hat{p}(t) < \frac{1+(1-t\lambda)\overline{B}(n)}{1+\overline{B}(n)}$$

where $\underline{B}(n) = \lambda m_1 \overline{k}\underline{s}(n), and\ \overline{B}(n) = \lambda m_1 \overline{k}\overline{s}(n).$

This result can be simply proved by using the relations (2.15),(2.16) and (3.5).

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