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On Sums of Range Quaternion Hermitian Matrices

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ABSTRACT:- Necessary and sufficient conditions for the sum of q-EP matrices are discussed. As an application it is shown that sum and sum of parallel summable q-EP matrices are discussed.

Keywords:- Moore - Penrose inverse, q-EP matrix, Sum of q-EP.

I. INTRODUCTION

Throught we shall deal with $n\times n$ quaternion matrices. Let A^* denote the conjugate transpose of A. Let A^- be the generalized inverse of A satisfying $AA^- A=A$ and z be the Moore-Penrose inverse of A[6]. Any matrix $A\in H_{nXn}$ is called q-EP[2] if $R(A) = R(A^*)$ and is called q-EP_r if A is q-EP and rk(A) = r, where N(A), R(A) and rk(A) denote the null space, range space and rank of A respectively. In this paper we study the question of when a sum of q-EP matrices are q-EP.

Theorem 1.1

Let A_i (i = 1 to m) be q-EP matrices. Then $A = \sum_{i=1}^{m} A_i$ is q-EP if any one of the following equivalent conditions hold:

(i) $N(A) \subseteq N(A_i)$ for each i,

(ii) $rk\begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ \cdots \\ A_m \end{pmatrix} = rk(A)$

Proof

(i) $\Rightarrow$ (ii) $N(A) \subseteq N(A_i)$ for each i=1 to m.

$\Rightarrow N(A) \subseteq \bigcap N(A_i)$

Since $N(A) = N(\sum A_i) \supseteq N(A_1) \bigcap N(A_2) \bigcap \ldots\ldots \bigcap N(A_m)$ it follows that

$N(A) \supseteq \bigcap N(A_i)$.

$$N(A) = \bigcap N(A_i) = N\begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ \cdots \\ A_m \end{pmatrix}$$

$$\text{rk(A)} = \text{rk}\begin{pmatrix} A_1 \\ A_2 \\ \dots \\ \dots \\ A_m \end{pmatrix}$$

Conversely, Since

$$N\begin{pmatrix} A_1 \\ A_2 \\ \dots \\ \dots \\ A_m \end{pmatrix} = \bigcap N(A_i) \subseteq N(A)$$

$$\text{rk}\begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ \dots \\ A_m \end{pmatrix} = \text{rk(A)}$$

$\Rightarrow$ N(A) = $\bigcap$ N(A_i).Hence N(A) $\subseteq$ N(A_i) for each i and (i) holds. Since each A_i is q-EP,
N(A_i) =N(A_i^*) for each i.
Now N(A) $\subseteq$N(A_i) for each i.
$\Rightarrow$ N(A)$\subseteq \bigcap$N(A_i)=$\bigcap$N(A_i^*)$\subseteq$N(A^*)
rk(A) = rk(A^*)
Hence N(A)=N(A^*)
Thus A is q-EP.
Hence the theorem.

Remark1.2

In particular, if A is non-singular then the conditions automatically hold and A is q-EP. Theorem(1.1) fails if we relax the conditions on A_i's.

Example 1.3

Consider A = $\begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$ is not q-EP.

B = $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is q-EP.

A+B = $\begin{pmatrix} 1 & 0 \\ k & 0 \end{pmatrix}$ is not q-EP. However,

N(A +B) $\subseteq$ N(A) , N(A+B) $\subseteq$ N(B) .

Moreover rk$\begin{bmatrix} A \\ B \end{bmatrix}$ = rk(A+B)

Example 1.4

If rank is additive, that is $rk(A) = \sum rk(A_i)$ then by Theorem 11 in [3] $R(A_i) \cap R(A_j)=\{0\}$, $i \neq j$, $N(A) \subseteq N(A_i)$ for each i .

Hence A is q-EP.The conditions given in therem(1) are weaker than the condition of rank additivity can been seen by the following example.

Example 1.5

Let $A = \begin{pmatrix} 1 & i \\ -i & 0 \end{pmatrix}$

$B = \begin{pmatrix} 1 & k \\ -k & 0 \end{pmatrix}$ are q-EP matrices

$A+B = \begin{pmatrix} 2 & i+k \\ -i-k & 0 \end{pmatrix}$ is q-EP.

Conditions (i) and (ii) in theorem(1.1) hold.

But $rk(A+B) \neq rk(A) + rk(B)$.

Theorem 1.6

Let A_i (i = 1 to m) be q-EP matrices such that $\sum_{i \neq j} A_i^* A_j = 0$ then $A = \sum A_i$ is q-EP.

Proof

Since $\sum_{i \neq j} A_i^* A_j = 0$

$$\begin{aligned} A^* A &= (\sum A_i)^* (\sum A_i) \\ &= (\sum A^*)(\sum A_i) \\ &= \sum A_i^* A_i \end{aligned}$$

$$N(A) = N(A^* A)$$

$$= N\left[\begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ \cdots \\ A_m \end{pmatrix}^* \begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ \cdots \\ A_m \end{pmatrix}\right]$$

$$= N\begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ \cdots \\ \cdots \\ A_m \end{bmatrix}$$

$$= N(A_1) \cap N(A_2) \cap \ldots\ldots \cap N(A_m)$$

$$= N(A_1^*) \cap N(A_2^*) \cap \ldots\ldots \cap N(A_m^*)$$

Hence $N(A) \subseteq N(A_i)$ for each i, Since each A_i is q-EP.A is q-EP, Now, A is q-EP follows from theorem(1.1).

Remarks 1.7

Theorem 1.6 fails if we relax the conditions that A_i's are EP, for instance,

$$A=\begin{pmatrix} 0 & 0 & 0 \\ 0 & -i-j-k & 0 \\ i+j+k & 0 & 0 \end{pmatrix}$$

$$B=\begin{pmatrix} 0 & -i-j-k & 0 \\ i+j+k & 0 & 0 \\ 0 & i+j+k & 0 \end{pmatrix}$$

$$A^*B=\begin{pmatrix} 0 & -(i+j+k)^2 & 0 \\ (i+j+k)^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B^*A=\begin{pmatrix} 0 & (i+j+k)^2 & 0 \\ -(i+j+k)^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ie, $A^*B+B^*A = 0$

Remarks 1.8

The conditions given in theorem(1.6) implies that theorem(1.1) but not conversely, this can been seen by the following example.

Example 1.9

Let $A=\begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix}$, $B=\begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$ Both A and B are q-EP matrices,

$N(A+B) \subseteq N(A)$, $N(A+B) \subseteq N(B)$. But $A^*B+B^*A \neq 0$.

Remarks 1.10

The conditions given in theorem(1.1) and theorem(1.6) are only sufficient for sum of q-EP matrices to be q-EP, but not necessary is illustrated by the following example.

Example 1.11

Let $A=\begin{pmatrix} 1 & i+j+k & 0 \\ -i-j-k & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 2 & i+j+k \\ 2 & 0 & 0 \\ -i-j-k & 0 & 0 \end{pmatrix}$ are q-EP.

Neither the conditions in theorem(1) and theorem(1.6) hold (A+B) is a q-EP.

Theorem 1.12

Let $A^*=H_1A$ and $B^*=H_2B$ Such that (H_1-H_2) is non-singular then (A+B) is q-EP if and only if $N(A+B) \subseteq N(B)$.

Proof

Since $A^*=H_1A$ and $B^*=H_2B$ by theorem(1.3) A and B are q-EP matrices. Since $N(A+B) \subseteq N(B)$. We can see that $N(A+B) \subseteq N(A)$.Hence by theorem(1.1).

$\therefore$ (A+B) is q-EP

Conversly, let us assume that (A+B) is q-EP by theorem in[3], there exists a non-singular matrix G such that

$(A+B)^* = G(A+B)$

$A^*+B^* = G(A+B)$

$H_1A+H_2B = G(A+B)$

$H_1A+H_2B = GA+GB$

$H_1A-GA = B-H_2B$

(H1-G)A = (G-H_2)B

L= MB where L = H_1G, M = G-H_2

(L+M)A = LA+MA

= MB+MA

= M(A+B)

(L+M)A = M(A+B) and

Similarly (L+M)B = L(A+B)

By hypothesis,

L+M = H_1-G+G-H_2

= H_1-H_2 is non-singular

N(A+B) $\subseteq$ (L(A+B)) , N(A+B) $\subseteq$ N(M(A+B))

= N ((L+M) = N((L+M)A)

= N(B) = N(A)

Therefore N(A+B) $\subseteq$ N(B)

Thus (A+B) is q-EP

$\Rightarrow$ N (A+B) $\subseteq$ N(A) and N(A+B) $\subseteq$ N(B)

Hence the theorem.

Remark 1.13

The condition (H_1-H_2) to be non-singular is essential in theorem (1.12) is illustrated in the following example.

Example 1.14

A = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ are both symmetric

Hence q-EP; H_1=H_2. A+B = $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is EP. But N(A+B) $\subseteq$ N(A) or N(B). Thus(1.12) fails.

II. PARALLEL SUMMABLE Q-EP MATRICES

Here, it is shown that sum and parallel sum of parallel summable q-EP matrices are q-EP.

Definition 1.15

A and B are said to be paralell summable(P.S) q-EP matrices if N(A+B) $\subseteq$ N(B) and $N(A+B)^* \subseteq N(B^*)$ (or) equivalently $N(A+B)^* \subseteq N(A^*)$.

Definition 1.16

If A and B are paralell summable q-EP then paralell sum of A and B denoted by B is defined as **A $\pm$ B =A(A+B)$^-$B.**

(The product A(A+B)$^-$B is invariant for all choices of generalized inverse $(A+B)^-$ of (A+B) under the conditions that A and B are parallel summable [6, page21]

Properties 1.17

Let A and B be a pair of parallel summable (P.S) q-EP matrices. Then the following hold:

P. 1 A $\pm$ B = B $\pm$ A

P. 2 A^* and B^* are p.s q-EP and (A $\pm$ B) = $A^* \pm B^*$

P. 3 If U is non singular then UA and UB are p.s q-EP and UA $\pm$ UB = U(A $\pm$ B)

P. 4 R(A $\pm$ B) = R(A) $\cap$ R(B) ; N(A $\pm$ B) = N(A) $\pm$ N(B)

P. 5 (A $\pm$ B) $\pm$ E = A $\pm$ (B $\pm$ E) if all the parallel sum operations involved are defined.

Lemma 1.18

Let A and B are q-EP matrices. Then A and B are P.S q-EP if and only if N(A+B) $\subseteq$ N(A).

Proof

A and B are p.s q-EP $\Rightarrow$ N(A + B) $\subseteq$ N (A) follows from Definition1.15.Conversely if N (A + B) $\subseteq$ N (A).

then N(A+B) $\subseteq$ N(B).

Since A and B are q-EP matrices by theorem(1.1) (A+B) is q-EP

Hence, $N(A+B)^* = N(A+B)$

$= N(A) \cap N(B)$

$= N(A)^* \cap N(B)^*$

Therefore, $N(A+B)^* \subseteq N(A_2^*)$ and $N(A+B)^* \subseteq N(B^*)$.By hypothesis, $N(A+B)^* \subseteq N(A)$. Hence A and B are P.S.Hence the theorem

Remark 1.19

Lemma 1 fails if we relax the condition that A and B are q-EP.

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \text{ is q-EP, } B = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix} \text{ is not q-EP.}$$

but $N(A+B)^* \not\subseteq N(A^*)$ or $N(A+B)^* \not\subseteq N(B^*)$.

Theorem 1.20

Let A and B be p. s q-EP matrices , then $(A \pm B)$ and $(A + B)$ are q-EP.

Proof

Since A and B are P.S q-EP matrices by Lemma 1.18,

$N(A+B) \subseteq N(A)$ and $N(A+B) \subseteq N(B)$.

Now, the fact (A+B) is q-EP follows from theorem1.1,

$R(A \pm B)^* = R(A)^* \pm R(B)^*$ **[By (P.2)]**

$= R((A)^* \cap R((B)^*)$ **[By (P.4)]**

$= R(A) \cap R(B)$ **[Since A and B are q-EP]**

$= R(A \pm B)$.

Thus $(A \pm B)$ is q-EP whenever A and B are q-EP.

Hence the theorem.

Remark 1.21

The sum of parallel sum of P.S q-EP matrices are q-EP.

Corollary 1.22

Let A and B be q-EP matrices such that $N(A+B) \subseteq N(B)$. If C is q-EP commuting with both A and B, then, $C(A+B)$ and $C(A \pm B) = CA \pm CB$ are q-EP.

Proof

A and B are q-EP . By Theorem**(1.1)**, (A + B) is q-EP.

Now A, B and (A + B) areq-EP. Since C commutes with A, B and (A + B),). By Theorem**1.20,**$CA \pm CB$ is q-EP. $C(A \pm B)$ is q-EP $\Rightarrow CA \pm CB$ is q-EP.

So,$C(A \pm B)$ is q-EP. Hence the corollary. ■

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