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On ranges and null spaces of a special type of operator named $\lambda - jection$. – Part II

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ABSTRACT: *In this article, $\lambda - jection$ has been introduced which is a generalization of trijection operator as introduced in P.Chandra's Ph. D. thesis titled "Investigation into the theory of operators and linear spaces" (Patna University,1977). We obtain relation between ranges and null spaces of two given $\lambda - jections$ under suitable conditions.*
Key Words: *projection, trijection, $\lambda - jection$*

I. Introduction

Dr. P. Chandra has defined a trijection operator in his Ph.D. thesis titled " Investigation into the theory of operators and linear spaces". [1]. A projection operator E on a linear space X is defined as $E^2 = E$ as given in Dunford and Schwartz [2] , p .37 and Rudin [3], p.126. In analogue to this , E is a trijection operator if $E^3 = E$. It is a generalization of projection operator in the sense that every projection is a trijection but a trijection is not necessarily a projection.

II. Definition

Let X be a linear space and E be a linear operator on X. We call E a $\lambda - jection$ if $E^3 + \lambda E^2 = (1+\lambda)E$, λ being a scalar. Thus if $\lambda = 0$, $E^3 = E$ *i.e.* E *is a* trijection. We see that $E^2 = E \Rightarrow E^3 = E$ and above condition is satisfied. Thus a projection is also a $\lambda - jection$.

III. Main Results

3.1 We first investigate the case when an expression of the form $aE^2 + bE$ is a projection where E is a $\lambda - jection$. For this we need

$(aE^2 + bE)^2 = aE^2 + bE$.

$\Rightarrow a^2E^4 + b^2E^2 + 2abE^3 = aE^2 + bE$ ……………………………………………………...(1)

From definition of $\lambda - jection$,

$E^3 = (1+\lambda)E - \lambda E^2$

so $E^4 = E.E^3 = (1+\lambda)E^2 - \lambda E^3$

$= (1+\lambda)E^2 - \lambda\{(1+\lambda)E - \lambda E^2\}$

$= (1+\lambda+\lambda^2)E^2 - \lambda(1+\lambda)E$

We put these values in (1) and after simplifying

$\{a^2(1+\lambda) + b^2 - a - \lambda(2ab - a^2\lambda)\}E^2$

$+\{(2ab - a^2\lambda)(1+\lambda) - b\}E = 0$

Equating Coefficients of E & E^2 to be 0, we get

$a^2(1+\lambda) + b^2 - a - \lambda(2ab - a^2\lambda) = 0$ ……………………………………………………(2)

$(2ab - a^2\lambda)(1+\lambda) - b = 0$ ………………………………………………………………..(3)

Adding (2) and (3), We get

$(2ab - a^2\lambda)(1+\lambda-\lambda) + a^2(1+\lambda) + b^2 - a - b = 0$

$\Rightarrow 2ab - a^2\lambda + a^2 + a^2\lambda + b^2 - a - b = 0$

$\Rightarrow a^2 + 2ab + b^2 - (a+b) = 0$

$\Rightarrow (a+b)^2 - (a+b) = 0$

$\Rightarrow (a+b)(a+b-1) = 0$

$\Rightarrow$Either $(a+b) = 0$ or $(a+b) = 1$

So for projection, the above two cases will be considered

Case (1): let $(a+b)=1$ then $b=1-a$

Putting the value of b = 1 - a in equation (2) , we get

$$a^2(1+\lambda+\lambda^2)-2a(1-a)\lambda-a+(1-a)^2=0$$

$\Rightarrow a^2(\lambda^2+3\lambda+2)-a(3+2\lambda)+1=0$

$\Rightarrow a^2(\lambda+1)(\lambda+2)-a(\lambda+1+\lambda+2)+1=0$

$\Rightarrow$ [a($\lambda+1$)-1] [a($\lambda+2$) $-$ 1]=0

$\Rightarrow a=\frac{1}{\lambda+1}\ or\ \frac{1}{\lambda+2}$

Then $b=\frac{\lambda}{\lambda+1}$ or $\frac{\lambda+1}{\lambda+2}$

Hence corresponding projections are

$\frac{E^2}{\lambda+1}+\frac{\lambda E}{\lambda+1}\ and\ \frac{E^2}{\lambda+2}+\frac{(\lambda+1)E}{\lambda+2}$

Case (2) :- Let a + b = 0 or b = $-a$

So from Equation (2)

$$a^2(1+\lambda)+a^2-a-\lambda\{-2a^2-a^2\lambda\}=0$$

$\Rightarrow a^2[\lambda^2+3\lambda+2]-a=0$

$\Rightarrow a[a(\lambda+1)(\lambda+2)-1]=0$

$\Rightarrow a=\frac{1}{(\lambda+1)(\lambda+2)}$; (Assuming $a\neq 0$)

Therefore, $b=\frac{-1}{(\lambda+1)(\lambda+2)}$

Hence the corresponding projection is

$$\frac{E^2-E}{(\lambda+1)(\lambda+2)}$$

So in all we get three projections. Call them A, B & C.

i.e. A= $\frac{E^2}{\lambda+2}+\frac{(1+\lambda)E}{\lambda+2}$, B = $\frac{E^2-E}{(\lambda+1)(\lambda+2)}$

and C = $\frac{E^2}{1+\lambda}+\frac{\lambda E}{1+\lambda}$.

3. 2 Relation between A, B and C.

A+B = $\frac{E^2}{\lambda+2}+\frac{(1+\lambda)E}{\lambda+2}+\frac{E^2}{(\lambda+1)(\lambda+2)}-\frac{E}{(\lambda+1)(\lambda+2)}$

$=\frac{E^2(\lambda+1)+(\lambda+1)^2E+E^2-E}{(\lambda+1)(\lambda+2)}$

$=\frac{E^2(\lambda+2)+\lambda(\lambda+2)E}{(\lambda+1)(\lambda+2)}$

$=\frac{E^2+\lambda E}{\lambda+1}$

$=\frac{E^2}{\lambda+1}+\frac{\lambda E}{\lambda+1}=C$

Hence $(A+B)^2=C^2\Rightarrow A^2+B^2+2AB=C^2$

$\Rightarrow A+B+2AB=C$

$\Rightarrow 2AB=0(Since\ \ A+B=C)$

$\Rightarrow AB=0$

Let $\mu=\lambda+1$

Then A = $\frac{E^2}{\mu+1}+\frac{\mu E}{\mu+1}, B=\frac{E^2-E}{\mu(\mu+1)}$ and $C=\frac{E^2}{\mu}+\frac{(\mu-1)E}{\mu}$

Also $A-\mu\beta=\frac{E^2}{\mu+1}+\frac{\mu E}{\mu+1}-\frac{E^2}{\mu+1}+\frac{E}{\mu+1}$

$$=\frac{\mu E+E}{\mu+1}=\frac{(\mu+1)E}{\mu+1}=E$$

Thus E = A □ μB.

3.3 On ranges and null spaces of λ-jection

We show that

$R_E = R_C$ and $N_E = N_C$

Where R_E stands for range of operator E and N_E for null space of E and similar notations for other operators.

Let $x \in R_E$ $then$ $x = Ez$ for $some$ z in X.

Therefore,

$$Cx = CEz = \frac{\{(\mu-1)E+E^2\}Ez}{\mu}$$
$$= \frac{\{(\mu-1)E^2+E^3\}z}{\mu}$$
$$= \frac{\{(\mu-1)E^2+\mu E-(\mu-1)E^2\}z}{\mu} \quad \text{(Since } E^3 = (1+\lambda)E - \lambda E^2\text{)}$$
$$= \left(\frac{\mu E}{\mu}\right)z = Ez = x$$

Thus $Cx = x \Rightarrow x \in R_C$

Therefore $R_E \subseteq R_C$

Again if $x \in R_C$ $then$ $x = Cx = \left(\frac{E^2+(\mu-1)E}{\mu}\right)x$

$$= \frac{E(E+\lambda I)x}{\lambda+1} \in R_E$$

Hence $R_C \subseteq R_E$

Therefore $R_E = R_C$

Now , z$\in N_E \Rightarrow Ez = 0$

$\Rightarrow \left(\frac{E^2+(\mu-1)E}{\mu}\right)z = 0$

$\Rightarrow Cz = 0$

$\Rightarrow$ z $\in N_C$

Therefore , $N_E \subseteq N_C$

Also if z$\in N_C \Rightarrow Cz = 0 \Rightarrow \left(\frac{E^2+(\mu-1)E}{\mu}\right)z = 0$

$$\Rightarrow E\left(\frac{E^2+(\mu-1)E}{\mu}\right)z = 0$$
$$\Rightarrow \left(\frac{E^3+(\mu-1)E^2}{\mu}\right)z = 0$$
$$\Rightarrow \left(\frac{\mu E}{\mu}\right)z = 0 \Rightarrow Ez = 0 \Rightarrow z \in N_E$$

Thus $N_c \subseteq N_E$,

Therefore , $N_E = N_C$

Now we show that

$\boldsymbol{R_A = \{z: Ez = z\}}$ $\boldsymbol{and}$ $\boldsymbol{R_B = \{z: Ez = -\mu z\}}$

Since A is a Projection ,

$R_A = \{z: Az = z\}$

Let $z \in R_A$. Then $Ez = EAz = E\left(\frac{E^2+\mu E}{\mu+1}\right)z$

$$= \left(\frac{E^3+\mu E^2}{\mu+1}\right)z$$
$$= \left(\frac{E^3+(\mu-1)E^2+E^2}{\mu+1}\right)z$$
$$= \left(\frac{\mu E+E^2}{\mu+1}\right)z$$

$= Az = z$

Thus $R_A \subseteq \{z: Ez = z\}$

Conversely , Let $Ez = z \; then \; E^2 z = z$

So $Az = \left(\frac{E^2+\mu E}{\mu+1}\right) z = \frac{z+\mu z}{\mu+1} = z \Rightarrow z \in R_A$

Hence $\{z: Ez = z\} \subseteq R_A$

Therefore, $R_A = \{z: Ez = z\}$

Next we show that

$R_B = \{z: Ez = -\mu z\}$

Since B is a Projection,

$R_B = \{z: Bz = z\}$

Let $Ez = -\mu z \; then \; E^2 z = \mu^2 z$

Hence $\left(\frac{E^2-E}{\mu(\mu+1)}\right)$ z $= \frac{\mu^2 z+\mu z}{\mu(\mu+1)} = \frac{\mu(\mu+1)}{\mu(\mu+1)} \; z = z$

i.e. $Bz = z$ $\left(since \; B = \frac{E^2-E}{\mu(\mu+1)}\right)$

$\Rightarrow z \in R_B$

$Therefore, \{z: Ez = -\mu z\} \subseteq R_B$

Conversely , let $z \subseteq R_B. Then \; Bz = z$

Hence $Ez = EBz = E\left(\frac{E^2-E}{\mu(\mu+1)}\right) z$

$= \left(\frac{E^3-E^2}{?(?+1)}\right) z$

But $E^3 - E^2 = \mu E - (\mu-1)E^2 - E^2$

$= \mu E - \mu E^2 = \mu(E - E^2)$

So $Ez = \frac{\mu(E-E^2)z}{\mu(\mu+1)} = \frac{-\mu(E^2-E)z}{\mu(\mu+1)}$

$= -\mu Bz = -\mu z$

$So \; R_B \subseteq \{z: Ez = -\mu z\}$

Therefore $R_B = \{z: Ez = -\mu z\}$

Now we show that $R_A \cap R_B = \{0\}$

Let $z \in R_A \cap R_B$

Then $z \in R_A \; and \; z \in R_B$

If $z \in R_A \; than \; Ez = z$

If $z \in R_B \; than \; Ez = -\mu z$

Thus $Ez = z = -\mu z$

$\Rightarrow \mu z + z = 0 \Rightarrow (\mu+1)z = 0 \Rightarrow z = 0 \; (since \; \mu + 1 \neq 0)$

Therefore , $R_A \cap R_B = \{0\}$

Theorem (1): Let $E_1, E_2 \; be \; commuting \; \lambda-jections \; on \; a \; linear \; space \; X$ Such that $R_{A_1} = R_{A_2}$

Then $E_1 E_2^2 = E_1^2 E_2$

Proof : $Let \; z \in X. \; Consider \; A_1 z \in R_{A_1}$

$Since \; R_{A_1} \subseteq R_{A_2}, A_1 z \in R_{A_2}$

Hence $E_2 A_1 z = A_1 z.$ $(since \; R_{A_2} = \{z: E_2 z = z\})$

Since $z \; is \; arbitrary, E_2 A_1 = A_1$

Also $R_{A_2} \subseteq R_{A_1}$ $, hence \; as \; above, E_1 A_2 = A_2$

$Now\ E_2A_1 = A_1 \Rightarrow E_2^2A_1 = E_2A_1$

$\Rightarrow E_2^2\left(\frac{E_1^2}{\mu+1} + \frac{\mu\ E_1}{\mu+1}\right) = E_2(\frac{E_1^2}{\mu+1} + \frac{\mu\ E_1}{\mu+1})$

$\Rightarrow E_2^2E_1^2 + \mu E_2^2E_1 = E_2E_1^2 + \mu E_2E_1$

$Since\ E_1, E_2\ Commute$,

$E_1^2E_2^2 + \mu E_1E_2^2 = E_1^2E_2 + \mu E_1E_2$

$\Rightarrow \mu E_1E_2^2 - E_1^2E_2 = \mu E_1E_2 - E_1^2E_2^2$ ……………………………………….(1)

$Also\ E_1A_2 = A_2 \Rightarrow E_1^2A_2 = E_1A_2$

$\Rightarrow E_1^2\left(\frac{E_2^2}{\mu+1} + \frac{\mu E_2}{\mu+1}\right) = E_1(\frac{E_2^2}{\mu+1} + \frac{\mu E_2}{\mu+1})$

$\Rightarrow E_1^2E_2^2 + \mu E_1^2E_2 = E_1E_2^2 + \mu E_1E_2$

$\Rightarrow \mu E_1^2E_2 - E_1E_2^2 = \mu E_1E_2 - E_1^2E_2^2$ ……………………….(2)

From (1) and (2), we get

$\mu E_1E_2^2 - E_1^2E_2 = \mu E_1^2E_2 - E_1E_2^2$

$\Rightarrow (\mu+1)E_1E_2^2 = (\mu+1)E_1^2E_2$

Since μ +1 ≠ 0 , $E_1E_2^2 = E_1^2E_2$.

Theorem (2): $Let\ E_1, E_2\ be\ commuting\ \lambda - Jections\ on$ $a\ linear\ Space\ X\ such\ shat\ R_{B_1} = R_{B_2}$, then $E_1E_2^2 = E_1^2E_2$

Proof : $Let\ z \in X.\ Consider\ B_1z \in R_{B_1}$

$Since\ R_{B_1} \subseteq R_{B_2}, B_1z \in R_{B_2}$

Hence $E_2(B_1z) = -\mu B_1z$

Since $z\ is\ arbitrary, E_2B_1 = -\mu B_1$

Also $R_{B_2} \subseteq R_{B_1}$ $, hence\ as\ above, E_1B_2 = -\mu B_2$

$Now\ E_2B_1 = -\mu B_1 \Rightarrow E_2^2B_1 = -\mu E_2B_1$

$\Rightarrow E_2^2\left(\frac{E_1^2 - E_1}{\mu(\mu+1)}\right) = -\mu E_2\left(\frac{E_1^2 - E_1}{\mu(\mu+1)}\right)$

$\Rightarrow E_2^2E_1^2 - E_2^2E_1 = -\mu E_2E_1^2 + \mu E_2E_1$

$Since\ E_1, E_2\ Commute,\ E_1^2E_2^2 - E_1E_2^2 = -\mu E_1^2E_2 + \mu E_1E_2$

$\Rightarrow \mu E_1^2E_2 - E_1E_2^2 = \mu E_1E_2 - E_1^2E_2^2$ …………………………………………….(1)

Also $E_1B_2 = -\mu B_2 \Rightarrow E_1^2B_2 = -\mu E_1B_2$

$Hence$ Interchanging E_1, E_2 we get as in(1)

$\mu E_1E_2^2 - E_1^2E_2 = \mu E_1E_2 - E_1^2E_2^2$ ……………………………………………..(2)

From (1) and (2) ,we get

$\mu E_1^2E_2 - E_1E_2^2 = \mu E_1E_2^2 - E_1^2E_2$

$\Rightarrow (\mu+1)E_1^2E_2 = (\mu+1)E_1E_2^2$

$Since\ \mu + 1 \neq 0, E_1E_2^2 = E_1^2E_2$

$\boldsymbol{Theorem}\ (\mathbf{3}): If\ E_1, E_2\ are\ commuting\ \lambda - Jections\ on\ a$ $linear\ space\ X\ such\ that\ R_{A_1} = R_{A_2}\ and\ R_{B_1} = R_{B_2}, than\ E_1 = E_2$

Proof : $Due\ to\ prev.theorems\ (1)\ and\ (2)$

$E_2A_1 = A_1\ and\ E_1A_2 = A_2, E_2B_1 = -\mu B_1$,

$E_1B_2 = -\mu B_2\ and\ E_1E_2^2 = E_1^2E_2$

Now,$E_2A_1 = A_1\ and\ E_2B_1 = -\mu B_1$

$\Rightarrow E_2A_1 - \mu E_2B_1 = A_1 + \mu^2B_1 = E_1^2$

$\Rightarrow E_2(A_1 - \mu B_1) = E_1^2 \Rightarrow E_2E_1 = E_1^2\ (Since\ A - \mu B = E)$

$Since E_1, E_2 Commute, E_1\ E_2 = E_1^2$. … … … … … … … … … … … … … … … … . (1)

$Also\ E_1A_2 = A_2 and\ E_1B_2 = -\mu B_2$

$\Rightarrow E_1A_2 - \mu E_1B_2 = A_2 + \mu^2 B_2 = E_2^2 \quad (since\ \ A_2 + \mu^2 B_2 = E_2^2)$

$\Rightarrow E_1(A_2 - \mu B_2) = E_2^2 \Rightarrow E_1E_2 \ = E_2^2$ ……………………………………..(2)

From (1) and (2) , we get $E_1^2 = E_2^2$

Hence $E_1E_2^2$=$E_1^2E_2 \Rightarrow E_1E_1^2 = E_2^2.E_2 \Rightarrow E_1^3 = E_2^3$

Therefore $E_1^3 + \lambda E_1^2 = E_2^3 + \lambda E_2^2$

$\Rightarrow (1+\lambda)E_1 = (1+\lambda)E_2$

$\Rightarrow E_1 = E_2 \ \ (since \ \ 1 + \lambda \neq 0).$

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