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On ranges and null spaces of a special type of operator named $\lambda - jection.$ – Part III

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ABSTRACT: *In this article, $\lambda - jection$ has been introduced which is a generalization of trijection operator as introduced in P.Chandra's Ph. D. thesis titled "Investigation into the theory of operators and linear spaces" (Patna University,1977). We obtain relation between ranges and null spaces of two given $\lambda - jections$ under suitable conditions.*

Key Words: *projection, trijection, $\lambda - jection$*

I. Introduction

Dr. P. Chandra has defined a trijection operator in his Ph.D. thesis titled " Investigation into the theory of operators and linear spaces". [1]. A projection operator E on a linear space X is defined as $E^2 = E$ as given in Dunford and Schwartz [2] , p .37 and Rudin, [3] p.126. In analogue to this, E is a trijection operator if $E^3 = E$. It is a generalization of projection operator in the sense that every projection is a trijection but a trijection is not necessarily a projection.

II. Definition

Let X be a linear space and E be a linear operator on X. We call E a $\lambda - jection$ if
$E^3 + \lambda E^2 = (1+\lambda)E, \lambda$ being a scalar. Thus if $\lambda = 0$
, $E^3 = E$ *i.e. E is a* trijection. We see that $E^2 = E \Rightarrow E^3 = E$ and above condition is satisfied. Thus a projection is also a $\lambda - jection$.

III. Main Results

3.1 We first investigate the case when an expression of the form $aE^2 + bE$ is a projection where E is a $\lambda - jection$. For this we need

$(aE^2 + bE)^2 = aE^2 + bE$.

$\Rightarrow a^2E^4 + b^2E^2 + 2abE^3 = aE^2 + bE$ ……………………………………………………..(1)

From definition of $\lambda - jection$,

$E^3 = (1+\lambda)E - \lambda E^2$

$so\ E^4 = E.E^3 = (1+\lambda)E^2 - \lambda E^3$

$= (1+\lambda)E^2 - \lambda\{(1+\lambda)E - \lambda E^2\}$

$= (1+\lambda+\lambda^2)E^2 - \lambda(1+\lambda)E$

We put these values in (1) and after simplifying

$\{a^2(1+\lambda) + b^2 - a - \lambda(2ab - a^2\lambda)\}E^2$

$+\{(2ab - a^2\lambda)(1+\lambda) - b\}E = 0$

Equating Coefficients of E & E^2 to be 0, we get

$a^2(1+\lambda) + b^2 - a - \lambda(2ab - a^2\lambda) = 0$ ……………………………………………………(2)

$(2ab - a^2\lambda)(1+\lambda) - b = 0$ ………………………………………………………………..(3)

Adding (2) and (3), We get

$(2ab - a^2\lambda)(1+\lambda-\lambda) + a^2(1+\lambda) + b^2 - a - b = 0$

$\Rightarrow 2ab - a^2\lambda + a^2 + a^2\lambda + b^2 - a - b = 0$

$\Rightarrow a^2 + 2ab + b^2 - (a+b) = 0$

$\Rightarrow (a+b)^2 - (a+b) = 0$

$\Rightarrow (a+b)(a+b-1) = 0$

$\Rightarrow$Either $(a+b) = 0$ or $(a+b) = 1$

So for projection, the above two cases will be considered

Case (1): let $(a+b)=1$ then $b=1-a$

Putting the value of b = 1 - a in equation (2) , we get

$$a^2(1+\lambda+\lambda^2)-2a(1-a)\lambda-a+(1-a)^2=0$$

$\Rightarrow a^2(\lambda^2+3\lambda+2)-a(3+2\lambda)+1=0$

$\Rightarrow a^2(\lambda+1)(\lambda+2)-a(\lambda+1+\lambda+2)+1=0$

$\Rightarrow$ [a($\lambda+1$)-1] [a($\lambda+2$) $-$ 1]=0

$\Rightarrow a=\frac{1}{\lambda+1}\ or\ \frac{1}{\lambda+2}$

Then $b=\frac{\lambda}{\lambda+1}$ or $\frac{\lambda+1}{\lambda+2}$

Hence corresponding projections are

$\frac{E^2}{\lambda+1}+\frac{\lambda E}{\lambda+1}\ and\ \frac{E^2}{\lambda+2}+\frac{(\lambda+1)E}{\lambda+2}$

Case (2) :- Let a + b = 0 or b = $-a$

So from Equation (2)

$$a^2(1+\lambda)+a^2-a-\lambda\{-2a^2-a^2\lambda\}=0$$

$\Rightarrow a^2[\lambda^2+3\lambda+2]-a=0$

$\Rightarrow a[a(\lambda+1)(\lambda+2)-1]=0$

$\Rightarrow a=\frac{1}{(\lambda+1)(\lambda+2)}$; (Assuming $a\neq 0$)

Therefore, $b=\frac{-1}{(\lambda+1)(\lambda+2)}$

Hence the corresponding projection is

$$\frac{E^2-E}{(\lambda+1)(\lambda+2)}$$

So in all we get three projections. Call them A, B & C.

i.e. A= $\frac{E^2}{\lambda+2}+\frac{(1+\lambda)E}{\lambda+2}$, B = $\frac{E^2-E}{(\lambda+1)(\lambda+2)}$

and C = $\frac{E^2}{1+\lambda}+\frac{\lambda E}{1+\lambda}$.

3.2 Relation between A, B, & C.

A+B = $\frac{E^2}{\lambda+2}+\frac{(1+\lambda)E}{\lambda+2}+\frac{E^2}{(\lambda+1)(\lambda+2)}-\frac{E}{(\lambda+1)(\lambda+2)}$

$=\frac{E^2(\lambda+1)+(\lambda+1)^2E+E^2-E}{(\lambda+1)(\lambda+2)}$

$=\frac{E^2(\lambda+2)+\lambda(\lambda+2)E}{(\lambda+1)(\lambda+2)}$

$=\frac{E^2+\lambda E}{\lambda+1}$

$=\frac{E^2}{\lambda+1}+\frac{\lambda E}{\lambda+1}=c$

Hence $(A+B)^2=C^2\Rightarrow A^2+B^2+2AB=C^2$

$\Rightarrow A+B+2AB=C$

$\Rightarrow 2AB=0(Since\ \ A+B=C)$

$\Rightarrow AB=0$

Let $\mu=\lambda+1$

Then A = $\frac{E^2}{\mu+1}+\frac{\mu E}{\mu+1}, B=\frac{E^2-E}{\mu(\mu+1)}$ and $C=\frac{E^2}{\mu}+\frac{(\mu-1)E}{\mu}$

Also $A-\mu\beta=\frac{E^2}{\mu+1}+\frac{\mu E}{\mu+1}-\frac{E^2}{\mu+1}+\frac{E}{\mu+1}$

$=\frac{\mu E+E}{\mu+1}=\frac{(\mu+1)E}{\mu+1}=E$

Thus $E=A-\mu B$.

3.3 On ranges and null spaces of $\boldsymbol{\lambda - jection}$

We show that

$R_E = R_C \;\; and \; N_E = N_C$

Where R_E stands for range of operator E and N_E for Null Space of E and similar notations for other operators.

Let $x \in R_E \; then \; x = Ez \;\; for \;\; some \; z \; in \; X.$

Therefore,

$$Cx = CEz = \frac{\{(\mu-1)E+E^2\}Ez}{\mu}$$
$$= \frac{\{(\mu-1)E^2+E^3\}\,Z}{\mu}$$
$$= \frac{\{(\mu-1)E^2+\mu E-(\mu-1)E^2\}\,Z}{\mu} \qquad \text{(Since } E^3 = (1+\lambda)E - \lambda E^2\text{)}$$
$$= \left(\frac{\mu E}{\mu}\right) z = Ez = x$$

Thus $Cx = x \Rightarrow x \in R_C$

Therefore $R_E \subseteq \; R_C$

Again if $x \in R_C \; then \; x = Cx = \left(\frac{E^2+(\mu-1)E}{\mu}\right) x$

$$= \frac{E(E+\lambda I)x}{\lambda+1} \in R_E$$

Hence $R_C \subseteq R_E$

Therefore $R_E = R_C$

Now , z$\in N_E \Rightarrow Ez = 0$

$\Rightarrow \left(\frac{E^2+(\mu-1)E}{\mu}\right) z = 0$

$\Rightarrow Cz = 0$

$\Rightarrow$ z $\in N_C$

Therefore , $N_E \subseteq \; N_C$

Also if z$\in N_C \Rightarrow \; Cz = 0 \Rightarrow \left(\frac{E^2+(\mu-1)E}{\mu}\right) z = 0$

$$\Rightarrow E\left(\frac{E^2+(\mu-1)E}{\mu}\right) z = 0$$
$$\Rightarrow \left(\frac{E^3+(\mu-1)E^2}{\mu}\right) z = 0$$
$$\Rightarrow \left(\frac{\mu E}{\mu}\right) z = 0 \Rightarrow Ez = 0 \;\; \Rightarrow z \in N_E$$

Thus $N_c \subseteq N_E$,

Therefore , $N_E = N_C$

Now we show that

$\boldsymbol{R_A = \{z: Ez = z\} \; and \; R_B = \{z: Ez = -\mu z\}}$

Since A is a Projection ,

$R_A = \{z: Az = z\}$

Let $z \in \; R_A$. Then $Ez = EAz = E\left(\frac{E^2+\mu E}{\mu+1}\right) z$

$$= \left(\frac{E^3+\mu E^2}{\mu+1}\right) z$$
$$= \left(\frac{E^3+(\mu-1)E^2+E^2}{\mu+1}\right) z$$
$$= \left(\frac{\mu E+E^2}{\mu+1}\right) z$$
$$= Az = z$$

Thus $R_A \subseteq \{z: Ez = z\}$

Conversely , Let $Ez = z \; then \; E^2 z = z$

So $Az = \left(\frac{E^2+\mu E}{\mu+1}\right) z = \frac{z+\mu z}{\mu+1} = z \Rightarrow z \in R_A$

Hence $\{z: Ez = z\} \subseteq R_A$
Therefore, $R_A = \{z: Ez = z\}$
Next we show
$R_B = \{z: Ez = -\mu z\}$
Since B is a Projection,
$R_B = \{z: Bz = z\}$
Let $Ez = -\mu z \; then \; E^2 z = \mu^2 z$
Hence $\left(\frac{E^2-E}{\mu(\mu+1)}\right)$ z $=\frac{\mu^2 z+\mu z}{\mu(\mu+1)} = \frac{\mu(\mu+1)}{\mu(\mu+1)} z = z$
i.e. $Bz = z$ $(since\ B = \frac{E^2-E}{\mu(\mu+1)})$
$\Rightarrow z \in R_B$
$Therefore, \{z: Ez = -\mu z\} \subseteq R_B$
Conversely , let $z \subseteq R_B . Then\ Bz = z$
Hence $Ez = EBz = E\left(\frac{E^2-E}{\mu(\mu+1)}\right)z$
$= \left(\frac{E^3-E^2}{?(?+1)}\right)z$
But $E^3 - E^2 = \mu E - (\mu - 1)E^2 - E^2$
$= \mu E - \mu E^2 = \mu(E - E^2)$
So $Ez = \frac{\mu(E-E^2)z}{\mu(\mu+1)} = \frac{-\mu(E^2-E)z}{\mu(\mu+1)}$
$= -\mu Bz = -\mu z$
$So\ R_B \subseteq \{z: Ez = -\mu z\}$
Therefore $R_B = \{z: Ez = -\mu z\}$
Now we show that $\boldsymbol{R_A} \cap \boldsymbol{R_B} = \{\boldsymbol{0}\}$
Let $z \in R_A \cap R_B$
Then $z \in R_A \; and \; z \in R_B$
If $z \in R_A \; then \; Ez = z$
If $z \in R_B \; then \; Ez = -\mu z$
Thus $Ez = z = -\mu z$
$\Rightarrow \mu z + z = 0 \Rightarrow (\mu + 1)z = 0 \Rightarrow z = 0 \; (since\ \mu + 1 \neq 0)$
Therefore , $R_A \cap R_B = \{0\}$
Theorem (1): If E_1, E_2 are commuting $\lambda - juction$ on a linear space X such that $R_{A_1} = R_{B_2}$ and $R_{A_2} = R_{B_1}$, then
$E_1 = \frac{-1}{\mu} E_1^2 E_2 \, , E_2 = \frac{-1}{\mu} E_1 E_2^2 \; and \; C_1 = C_2$
Proof : Given $R_{A_1} \subseteq R_{B_2}$
Let $z \in X, then\ A_1 z \in R_{A_1} \Rightarrow A_1 Z \in R_{B_2} \Rightarrow E_2(A_1 z) = -\mu A_1 z$
Since z is arbitrary, So $E_2 A_1 = -\mu A_1,$
Again, given that $R_{B_2} \subseteq R_{A_1}$.
Now $B_2 z \in R_{B_2} \Rightarrow B_2 z \in R_{A_1} \Rightarrow E_1 B_2 z = B_2 z$
Since z is arbitrary , $E_1 B_2 = B_2$
Hence we have $E_2 A_1 = -\mu A_1 \; and \; E_1 B_2 = B_2$
Similarly , $R_{A_2} = R_{B_1} \Rightarrow E_1 A_2 = -\mu A_2 \; and \; E_2 B_1 = B_1$
Thus $E_2 A_1 == -\mu A_1 \; and \; E_2 B_1 = B_1$
Hence $E_2(A_1 - \mu B_1) = -\mu A_1 - \mu B_1 = -\mu(A_1 + B_1) = -\mu C_1$
$\Rightarrow E_2 E_1 = -\mu C_1$ or $E_1 E_2 = -\mu C_1 \; (Since\ E_1, E_2 Commute)$
Also $E_1 A_2 = -\mu A_2$ and $E_1 B_2 = B_2$
Therefore $E_1(A_2 - \mu B_2) = -\mu A_2 - \mu B_2 = -\mu C_2$
$\Rightarrow E_1 E_2 = -\mu C_2 = -\mu C_1$

$\Rightarrow C_1 = C_2$

Also $-\frac{1}{\mu} E_1^2 E_2 = \frac{-1}{\mu} E_1(E_1E_2) = \frac{-1}{\mu} E_1(-\mu C_1) = E_1 C_1 = E_1$

And $\frac{-1}{\mu} \; E_1 E_2^2 = \frac{-1}{\mu}(E_1E_2)E_2 = \frac{-1}{\mu}(-\mu C_2)E_2 = C_2E_2 = E_2$

Theorem (2): if E_1 *and* E_2 are two commuting $\lambda - juctions$ on a linear space X (with $\lambda \neq 0$) such that $R_{A_1} = N_{E_2}$ and $R_{A_2} = N_{E_1}$, then

$$E_1^2 E_2 = E_1 E_2^2 \text{ and } E_1 - E_2 = E_1^2 - E_2^2$$

Proof : Let $z \in X$ *then* $A_1 z \in R_{A_1} \subseteq N_{E_2}$

$\Rightarrow A_1 z \in N_{E_2} \Rightarrow E_2(A_1 z) = 0, \forall z$

$\Rightarrow E_2 A_1 = 0$

$E_2(I - C_2)z = (E_2 - E_2C_2)z = (E_2 - E_2)z = 0$

$\Rightarrow (I - C_2)z \in N_{E_2} \subseteq \; R_{A_1} \quad \Rightarrow A_1(I - C_2)z = (I - C_2)z, \forall z$

$\Rightarrow A_1(I - C_2) = I - C_2 \Rightarrow A_1 - A_1C_2 = I - C_2 \; \ldots. (1)$

Similarly $R_{A_2} = N_{E_1}$, *So interchanging suffixes* 1 *and* 2,

$E_1 A_2 = 0$ *and* $A_2 - A_2C_1 = I - C_1 \quad \ldots\ldots. \; (2)$

Now $E_2A_1 - E_2A_2 = 0$

$\Rightarrow E_2\left(\frac{E_1^2+\mu E_1}{\mu+1}\right) - E_1\left(\frac{E_2^2+\mu E_2}{\mu+1}\right) = 0$

$\Rightarrow E_2E_1^2 + \mu E_2E_1 - (E_1E_2^2 + \mu E_1E_2) = 0$

Since E_1, E_2 *commute*, $E_1^2E_2 + \mu E_1E_2 - E_1E_2^2 - \mu E_1E_2 = 0$

$\Rightarrow E_1^2E_2 - E_1E_2^2 = 0 \Rightarrow E_1^2E_2 = E_1E_2^2$

Now subtracting equation (2) from (1), we get

$A_1 - A_2 - (A_1C_2 - A_2C_1) = C_1 - C_2 \quad \ldots\ldots\ldots\ldots\ldots (3)$

But $A_1C_2 - A_2C_1 = \left(\frac{E_1^2+\mu E_1}{\mu+1}\right)\left(\frac{E_2^2+(\mu-1)E_2}{\mu}\right) - \left(\frac{E_2^2+\mu E_2}{\mu+1}\right)\left(\frac{E_1^2+(\mu-1)E_1}{\mu}\right)$

$$= \frac{E_1^2E_2^2 + (\mu-1)E_1^2E_2 + \mu E_1E_2^2 + \mu(\mu-1)E_1E_2 - \{E_2^2E_1^2 + (\mu-1)E_2^2E_1 + \mu E_2E_1^2 + \mu(\mu-1)E_2E_1\}}{\mu(\mu+1)}$$

$$= \frac{E_1^2E_2^2+(\mu-1)E_1^2E_2+\mu E_1E_2^2+\mu(\mu-1)E_1E_2-E_2^2E_1^2+(\mu-1)E_2^2E_1 - \mu E_2E_1^2-\mu(\mu-1)E_2E_1}{\mu(\mu+1)}$$

$$= \frac{(\mu-1)E_1^2E_2 + \mu E_1E_2^2 - (\mu-1)E_2^2E_1 - \mu E_2E_1^2}{\mu(\mu+1)}$$

$= 0$ (*Since* E_1, E_2 *are commuting* $\lambda - jections$ *and* $E_1^2E_2 = E_1E_2^2$) $\ldots \ldots\ldots\ldots\ldots\ldots\ldots..$ (4)

Hence from (3) and (4), we have

$A_1 - A_2 = C_1 - C_2 \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (5)$

$\Rightarrow \frac{E_1^2+\mu E_1}{\mu+1} - \frac{E_2^2+\mu E_2}{\mu+1} = \frac{E_1^2+(\mu-1)E_1}{\mu} - \frac{E_2^2+(\mu-1)E_1}{\mu}$

$\Rightarrow \frac{(E_1^2-E_2^2)+\mu(E_1-E_2)}{\mu+1} = \frac{E_1^2-E_2^2+(\mu-1)(E_1-E_2)}{\mu}$

$\Rightarrow \mu(E_1^2 - E_2^2) + \mu^2(E_1 - E_2) = (\mu + 1(E_1^2 - E_2^2) + (\mu^2 - 1)\,(E_1 - E_2)$

$\Rightarrow (\mu^2 - \mu^2 + 1)(E_1 - E_2) = (\mu + 1 - \mu)(E_1^2 - E_2^2)$

$\Rightarrow E_1 - E_2 = E_1^2 - E_2^2$

Theorem (3): If E_1, E_2 *are two commuting* $\lambda - jections$ *on a linear* Space $X \; s.t.\; R_{B_1} = N_{E_2}$ and $R_{B_2} = N_{E_1}$ then

$E_1E_2^2 = E_1^2E_2$ *and* $\mu E_1 + E_1^2 = \mu E_2 + E_2^2$

Proof : Let $z \in X$, *then* $B_1 z \in R_{B_1} \subseteq N_{E_2}$

$\Rightarrow E_2B_1z = 0, \forall z \Rightarrow E_2B_1 = 0$

Also $(I - C_2)z \in N_{E_2} \subseteq R_{B_1} \Rightarrow E_1(I - C_2)z = -\mu(I - C_2)z, \forall z$

$\Rightarrow E_1(I - C_2) = -\mu(I - C_2) \Rightarrow E_1 - E_1C_2 = -\mu I + \mu C_2$ …. ……………………………….(1)

Siamiliarly $R_{B_2} = N_{E_1}$

$\Rightarrow E_1B_2 = 0, E_2 - E_2C_1 = -\mu I + \mu C_1$ ………….……………………………………………..(2)

Now , $E_2B_1 = 0 \Rightarrow E_2B_1 = E_2\left(\frac{E_1^2 - E_1}{\mu(\mu+1)}\right) = \frac{E_2E_1^2 - E_2E_1}{\mu(\mu+1)} = 0$

Hence $E_2E_1^2 = E_2E_1$

Since E_1, E_2 commute, we have $E_1^2E_2 = E_1\,E_2$ …………………………………………….. (3)

Similarly , $E_1B_2 = 0 \Rightarrow E_1E_2^2 = E_1E_2$ … (4)

Hence from equations (3) and (4)

$E_1E_2^2 = E_1^2E_2 (= E_1E_2)$ ………………………………………………………………….(5)

Subtracting , equations (2) from (1) we have

$E_1 - E_2 - (E_1C_2 - E_2C_1) = \mu(C_2 - C_1)$

But $E_1C_2 - E_2C_1 = E_1\left(\frac{E_2^2 + (\mu-1)E_2}{\mu}\right) - E_2\left(\frac{E_1^2 + (\mu-1)E_1}{\mu}\right)$

$= \frac{E_1E_2^2 - E_2E_1^2 + (\mu-1)(E_1E_2 - E_2E_1)}{\mu} = 0$

(using equation (5) and using the fact that $E_1, E_2\ commute$)

So $E_1 - E_2 = \mu(C_2 - C_1) = \mu\left\{\frac{E_2^2 + (\mu-1)E_2}{\mu} - \frac{E_1^2 + (\mu-1)E_2}{\mu}\right\}$

$= E_2^2 + (\mu - 1)E_2 - E_1^2 - (\mu - 1)E_1$

$= E_2^2 - E_1^2 + (\mu - 1)(E_2 - E_1)$

$\Rightarrow \mu(E_1 - E_2) = E_2^2 - E_1^2$

$\Rightarrow \mu E_1 + E_1^2 = \mu E_2 + E_2^2$ Proved

Theorem (4): If $E_1 and\ E_2$ are two commuting $\lambda - jections$ on a linear space $X\ s.t. N_{E_1} = R_{E_2}\ and\ N_{E_2} = R_{E_1}\ then\ E_1E_2 = 0\ and$

$$\mu I - (\mu - 1)(E_1 + E_2) = E_1^2 + E_2^2$$

Proof : Let $z \epsilon X\ then\ E_1 z \epsilon R_{E_1} \subseteq N_{E_2}$

$\Rightarrow E_2E_1z = 0 \Rightarrow E_2E_1 = 0 \Rightarrow E_1E_2 = 0$

Now $(I - C_2)z \epsilon N_{E_2} \subseteq R_{E_1} \Rightarrow (I - C_2)z \epsilon R_{C_1}$

$\Rightarrow C_1(I - C_2)z = (I - C_2)z\ ; \forall z \in X$

$\Rightarrow C_1 - C_1C_2 = I - C_2$

But $C_1C_2 = \frac{(\mu-1)E_1 + E_1^2}{\mu} \times \frac{(\mu-1)E_2 + E_2^2}{\mu}$

$= \frac{(\mu-1)^2E_1E_2 + (\mu-1)E_1E_2^2 + (\mu-1)E_1^2E_2 + E_1^2E_2^2}{\mu^2} = 0\ \ (Since\ E_1E_2 = 0)$

$Therfore,\ C_1 = I - C_2 \Rightarrow C_1 + C_2 = I$

$\Rightarrow \frac{(\mu-1)E_1 + E_1^2}{\mu} + \frac{(\mu-1)E_2 + E_2^2}{\mu} = I$

$\Rightarrow (\mu - 1)(E_1 + E_2) + E_1^2 + E_2^2 = \mu I$

$\Rightarrow \mu I - (\mu - 1)(E_1 + E_2) = E_1^2 + E_2^2$ Proved

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