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On some locally closed sets and spaces in Ideal Topological Spaces

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ABSTRACT: *In this paper we introduce and characterize some new generalized locally closed sets known as $\hat{\delta}_s$-locally closed sets and spaces are known as $\hat{\delta}_s$-normal space and $\hat{\delta}_s$-connected space and discussed some of their properties.*

Keywords and Phrases: *$\hat{\delta}_s$-locally closed sets, $\hat{\delta}_s$-normal space, $\hat{\delta}_s$-connected space.*

I. Introduction

In topological spaces locally closed sets were studied more by Bourbaki [2] in 1966, which is the intersection of an open set and a closed set. Kuratowski [4] was introduced the local function in ideal spaces. Vaidyanathaswamy [10] was given much importance to the topic and ideal topological space. Balachandran, Sundaram and Maki [1] introduced and investigated the concept of generalized locally closed sets. Navaneethakrishnan and Sivaraj [7] were introduced the concept of Ig-locally*-closed sets in ideal topological spaces. Navaneethakrishnan, Paulraj Joseph and Sivaraj [8] introduced and investigated the concept of Ig-normal and Ig-regular spaces. The purpose of this paper is to introduce and study the notions of locally closed sets, normal space and connectedness in Ideal topological spaces. We study the notions of $\hat{\delta}_s$-locally closed sets, $\hat{\delta}_s$-normal space, $\hat{\delta}_s$-separated sets and $\hat{\delta}_s$-connectedness in ideal topological spaces.

II. Preliminaries

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be
(i) a generalized closed (briefly g-closed) set [5] if $cl(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ).
(ii) a generalized locally closed (briefly GLC) set [1] if $A = U \cap F$ where U is g-open and F is g-closed in (X, τ).

Definition 2.2. For a subset A of (X, τ).
(i) $A \subseteq GLC^*(X, \tau)$ [1] if there exist a g-open set U and a closed set F of (X, τ) such that $A = U \cap F$.
(ii) $A \in GLC^{**}(X, \tau)$ [1] if there exist an open set U and a g-closed set F of (X, τ) such that $A = U \cap F$.

Definition 2.3. A topological space (X, τ) is said to be $T_{½}$-space [5] if every g-closed set in it is closed.
Definition 2.4. Let (X, τ, I) be an ideal space. A subset A is said to be
(i) Ig-closed [3] if $A^* \subset U$ whenever $A \subset U$ and U is open.
(ii) Ig-locally *-closed [7] if there exist an Ig-open set U and a *-closed set F such that $A = U \cap F$.

Notation 2.5. The class of all Ig-locally *-closed sets in (X, τ, I) is denoted by IgLC (X, τ, I) or simply IgLC.

Definition 2.6. [11] Let (X, τ, I) be an ideal topological space, A a subset of X and x a point of X.
(i) x is called a δ-I-cluster point of A if $A \cap int(cl^*(U)) \neq \phi$ for each open neighborhood of x.
(ii) The family of all δ-I-cluster points of A is called the δ-I-closure of A and is denoted by $[A]_{\delta\text{-}I}$ and
(iii) A subset A is said to be δ-I-closed if $[A]_{\delta\text{-}I} = A$. The complement of a δ-I-closed set of X is said to be δ-I-open.

Remark 2.7. [9] From Definition 2.6 it is clear that $[A]_{\delta\text{-}I} = \{x \in X : int(cl^*(U)) \cap A \neq \phi, \text{ for each } U \in \tau(x)\}$.
Notation 2.8. [9] $[A]_{\delta\text{-}I}$ is denoted by σcl(A).

Definition 2.9. [9] Let (X, τ, I) be an ideal space. A subset A of X is said to be $\hat{\delta}_s$-closed if σcl(A) ⊂ U whenever A⊂U and U is semi-open.

Lemma 2.10. [11] $\tau_s \subset \tau_{\delta\text{-}I} \subset \tau$.

Remark 2.11. [11] τ_s and $\tau_{\delta\text{-}I}$ are topologies formed by δ-open sets and δ-I-open sets respectively.

Lemma 2.12. [9] Intersection of a $\hat{\delta}_s$-closed and δ-I-closed set is $\hat{\delta}_s$-closed.

Lemma 2.13. [9] σcl(A) = {x∈X : int(cl*(U)) ∩ A ≠ ϕ, for all U∈τ(x)} is closed.

Remark 2.14. 1.[5] It is true that every closed set is g-closed but not conversely

2. [3] every g-closed set is Ig-closed but not conversely.
3. [9] every δ-I-closed set is $\hat{\delta}_s$-closed but not conversely.
4. [11] every δ-I-closed set is closed but not conversely.

III. $\hat{\delta}_S$-LOCALLY CLOSED SETS

In this section we introduce and study a new class of generalized locally closed set in an ideal topological space (X, τ, I) known as $\hat{\delta}_s$-locally closed sets.

Definition 3.1. A subset A of an ideal topological space (X, τ, I) is called $\hat{\delta}_s$-locally closed set (briefly $\hat{\delta}_s$lc) if A = U ∩ F where U is $\hat{\delta}_s$-open and F is $\hat{\delta}_s$-closed in (X, τ, I).

Notation 3.2. The class of all $\hat{\delta}_s$-locally closed sets in (X, τ, I) is denoted by $\hat{\delta}_s$LC (X, τ, I) or simply $\hat{\delta}_s$LC.

Definition 3.3. For a subset A of (X, τ, I), A∈ $\hat{\delta}_s$LC* (X, τ, I) if there exist a $\hat{\delta}_s$-open set U and a closed set F of (X, τ, I) such that A = U ∩ F.

Definition 3.4. For a subset A of (X, τ, I), A∈ $\hat{\delta}_s$LC** (X, τ, I) if there exist an open set U and a $\hat{\delta}_s$-closed set F of (X, τ, I) such that A = U ∩ F.

Proposition 3.5. Let A be a subset of an ideal space (X, τ, I). Then the following holds.

(i) If A∈ $\hat{\delta}_s$LC, then A∈GLC

(ii) If A∈ $\hat{\delta}_s$LC*, then A∈GLC, A∈GLC*, A∈I_gLC

(iii) If A ∈ $\hat{\delta}_s$LC**, then A∈GLC**, A∈GLC

Proof. The proof follows from the Remark 2.14 and Definitions.

Remark 3.6. The following examples shows that the converse of the above proposition is not always true.

Example 3.7. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}} and I = {ϕ, {c}}. Let A = {a, b, c}. Then A∈GLC, GLC*, GLC** but not in $\hat{\delta}_s$LC, $\hat{\delta}_s$LC*.

Example 3.8. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}} and I = {ϕ, {b}, {d}, {b, d}}. Let A = {a, c}. Then A∈GLC, GLC*, IgLC but not in $\hat{\delta}_s$LC**.

Theorem 3.9. For a subset A of an Ideal Space (X, τ, I). A ∈ $\hat{\delta}_s$LC* if and only if A = U ∩cl(A) for some $\hat{\delta}_s$-open set U.

Proof. Necessity - Let A∈ $\hat{\delta}_s$LC*, then there exist a $\hat{\delta}_s$-open set U and a closed set F in (X, τ, I) such that A = U ∩ F. Since A ⊆ U and A ⊆ cl(A), we have A ⊆ U ∩ cl(A). Conversely, since A ⊆ F, cl(A) ⊆ cl(F). Since F is closed, cl(F) = F. Therefore, cl(A) ⊆ F and A = U ∩ F ⊇ U ∩ cl(A). Hence A = U ∩ cl(A).

Sufficiency - Since U is $\hat{\delta}_s$-open and cl(A) is closed we have U ∩ cl(A) ∈ $\hat{\delta}_s$LC*.

Theorem 3.10. For a subset A of an Ideal Space (X, τ, I). cl(A) – A is $\hat{\delta}_s$-closed, if and only if A $\cup$ (X–cl(A)) is $\hat{\delta}_s$-open.

Proof. Necessity - Let F = cl(A)–A. By hypothesis, F is $\hat{\delta}_s$-closed and X–F = X $\cap$ (X–F) = X $\cap$ (X–(cl(A) –A)) = A $\cup$ (X–cl(A)). Since X–F is $\hat{\delta}_s$-open, A $\cup$ (X–cl(A)) is $\hat{\delta}_s$-open.
Sufficiency - Let U = A $\cup$(X–cl(A)). By hypothesis U is $\hat{\delta}_s$-open. Then X–U is $\hat{\delta}_s$-closed and X–U = X–(A$\cup$(X–cl(A)) = cl(A) $\cap$ (X–A) = cl(A) –A. Hence proved.

Definition 3.11. [9] The intersection of all $\hat{\delta}_s$-closed subset of (X, τ, I) that contains A is called $\hat{\delta}_s$-closure of A and it is denoted by $\hat{\delta}_s$cl(A). That is $\hat{\delta}_s$cl(A) = $\cap$ {F:A$\subset$F, F is $\hat{\delta}_s$-closed}. $\hat{\delta}_s$cl(A) is always $\hat{\delta}_s$-closed.

Theorem 3.12. For a subset A of an Ideal Space (X, τ, I), the following are equivalent.
(i) A $\in$ $\hat{\delta}_s$LC
(ii) A = U $\cap$ $\hat{\delta}_s$cl(A) for some $\hat{\delta}_s$-open set U
(iii) $\hat{\delta}_s$cl(A) – A is $\hat{\delta}_s$-closed
(iv) A $\cup$ (X–$\hat{\delta}_s$cl(A)) is $\hat{\delta}_s$-open.

Proof. (i) $\Rightarrow$ (ii). Let A$\in$ $\hat{\delta}_s$LC, then there exist a $\hat{\delta}_s$-open set U and a $\hat{\delta}_s$-closed set F in (X, τ, I) such that A = U $\cap$ F. Since A $\subseteq$ U and A $\subseteq$ $\hat{\delta}_s$cl(A), we have A $\subseteq$ U $\cap$ $\hat{\delta}_s$cl(A). Conversely, since A $\subseteq$ F, $\hat{\delta}_s$cl(A) $\subseteq$ $\hat{\delta}_s$-cl(F). Since F is $\hat{\delta}_s$-closed, $\hat{\delta}_s$cl(F) = F. Therefore, $\hat{\delta}_s$cl(A) $\subseteq$ F and A = U$\cap$F$\supseteq$U$\cap$ $\hat{\delta}_s$cl(A). This proves (ii)
(ii) $\Rightarrow$ (i). Since U is $\hat{\delta}_s$-open and $\hat{\delta}_s$cl(A) is $\hat{\delta}_s$-closed we have U $\cap$ $\hat{\delta}_s$cl(A) $\in$ $\hat{\delta}_s$LC.
(iii) $\Rightarrow$ (iv). Let F = $\hat{\delta}_s$cl(A)–A. By hypothesis, F is $\hat{\delta}_s$-closed and X–F = X $\cap$ (X–F) = X $\cap$ (X–($\hat{\delta}_s$cl(A) –A)) = A $\cup$ (X–$\hat{\delta}_s$cl(A)). Since X–F is $\hat{\delta}_s$-open, A $\cup$ (X–$\hat{\delta}_s$cl(A)) is $\hat{\delta}_s$-open.
(iv) $\Rightarrow$ (iii). Let U = A $\cup$(X–$\hat{\delta}_s$cl(A)). By hypothesis U is $\hat{\delta}_s$-open. Then X–U is $\hat{\delta}_s$-closed and X–U = X–(A$\cup$(X–$\hat{\delta}_s$cl(A)) = $\hat{\delta}_s$cl(A) $\cap$ (X–A) = $\hat{\delta}_s$cl(A) –A. This proves (iii).
(ii) $\Rightarrow$ (iv). Let A = U $\cap$ $\hat{\delta}_s$cl(A) for some $\hat{\delta}_s$-open set U. Now, A $\cup$(X–$\hat{\delta}_s$cl(A)) = (U$\cap$$\hat{\delta}_s$cl(A)) $\cup$ (X–$\hat{\delta}_s$cl(A)) = (U$\cup$X–$\hat{\delta}_s$cl(A)) $\cap$ ($\hat{\delta}_s$cl(A) $\cup$ (X–$\hat{\delta}_s$cl(A)) = (U$\cup$X–$\hat{\delta}_s$cl(A)) $\cap$ X = (U $\cup$ (X–$\hat{\delta}_s$cl(A))) is $\hat{\delta}_s$-open.
(iv) $\Rightarrow$ (ii) Let U = A $\cup$(X–$\hat{\delta}_s$cl(A)). Then U is $\hat{\delta}_s$-open. Now, U $\cap$ $\hat{\delta}_s$cl(A) = (A$\cup$(X–$\hat{\delta}_s$cl(A))) $\cap$ $\hat{\delta}_s$cl(A) = ($\hat{\delta}_s$cl(A) $\cap$A) $\cup$ ($\hat{\delta}_s$cl(A) $\cap$ (X–$\hat{\delta}_s$cl(A))) = A $\cup$ ϕ = A. Therefore A = U $\cap$ $\hat{\delta}_s$cl(A) for some $\hat{\delta}_s$-open set U.

Theorem 3.13. For a subset A of (X, τ, I). If A$\in$ $\hat{\delta}_s$LC** then there exist an open set U such that A = U $\cap$ $\hat{\delta}_s$cl(A).

Proof. Let A$\in$ $\hat{\delta}_s$LC**. Then there exists an open set U and a $\hat{\delta}_s$-closed set F in (X, τ, I) such that A = U$\cap$F. Since A$\subseteq$U and A$\subseteq$$\hat{\delta}_s$cl(A), we have A$\subseteq$U$\cap$$\hat{\delta}_s$cl(A). Conversely, since A$\subseteq$F, $\hat{\delta}_s$cl(A) $\subseteq$ $\hat{\delta}_s$cl(F). But $\hat{\delta}_s$cl(F)=F, since F is $\hat{\delta}_s$-closed. Therefore, $\hat{\delta}_s$cl(A)$\subseteq$F and A=U$\cap$F $\supseteq$U$\cap$$\hat{\delta}_s$cl(A).

Theorem 3.14. Let A and B be any two subsets of (X, τ, I). If A$\in$ $\hat{\delta}_s$LC* and B is closed, then A$\cap$B$\in$ $\hat{\delta}_s$LC*.

Proof. If A$\in$ $\hat{\delta}_s$LC* then there exists a $\hat{\delta}_s$-open set U and a closed set F in (X, τ, I) such that A=U$\cap$F. Now, A$\cap$B = (U$\cap$F) $\cap$B = U$\cap$(F$\cap$B) $\in$ $\hat{\delta}_s$LC*.

Theorem 3.15. Let A and B be two subsets of (X, τ, I). If A$\in$ $\hat{\delta}_s$LC** and B is open then A$\cap$B$\in$ $\hat{\delta}_s$LC**.

Proof. If A$\in$ $\hat{\delta}_s$LC**, then there exist an open set U and a $\hat{\delta}_s$-closed set F such that A = U$\cap$F. Then A$\cap$B=(U$\cap$F)$\cap$B = (U$\cap$B) $\cap$F $\in$ $\hat{\delta}_s$LC**.

Since X is open, closed, $\hat{\delta}_s$-open and $\hat{\delta}_s$-closed, a $\hat{\delta}_s$-closed subset A of X belongs to $\hat{\delta}_s$LC and $\hat{\delta}_s$LC**. A $\hat{\delta}_s$-open subset B of X belongs to $\hat{\delta}_s$LC and $\hat{\delta}_s$LC*. A closed subset C of X belongs to $\hat{\delta}_s$LC* and an open subset of X belongs to $\hat{\delta}_s$LC**. The following examples shows that the converse of all are not always true.

Example 3.16. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d} and I = {ϕ, {b}, {c}, {b,c}. Let A = {b,c}. Then A$\in\hat{\delta}_s$LC and $\hat{\delta}_s$LC** but A is not $\hat{\delta}_s$-closed.

Example 3.17. Let X and τ as in Example 3.16 and I = {ϕ, {a}, {b}, {a,b}. Let A = {a, c}. Then A$\in\hat{\delta}_s$LC and $\hat{\delta}_s$LC* but A is not $\hat{\delta}_s$-open.

Example 3.18. Let X and τ as in Example 3.16 and I = {ϕ, {a}, {d}, {a,d}. Let A = {b, c}. Then A$\in\hat{\delta}_s$LC* but A is not closed.

Example 3.19. In Example 3.7. Let A = {a, d}. Then A $\in\hat{\delta}_s$LC** but A is not open.

Theorem 3.20. Let (X, τ, I) be $T_{½}$-space. If A is $\hat{\delta}_s$-closed, then A$\in\hat{\delta}_s$LC*.

Proof. Let (X, τ, I) be a $T_{½}$-space and A be a $\hat{\delta}_s$-closed set. Since every $\hat{\delta}_s$-closed set is g-closed, A is g-closed. By hypothesis, A is closed and hence A$\in\hat{\delta}_s$LC*.

Theorem 3.21. Let (X, τ, I) be an ideal space and A, B are subsets of X. Then the following hold.

(i) If A, B $\in\hat{\delta}_s$LC* then A $\cap$B $\in\hat{\delta}_s$LC*.

(ii) If A, B$\in\hat{\delta}_s$LC then A$\cap$B $\in\hat{\delta}_s$LC.

(iii) If A, B $\in\hat{\delta}_s$LC** then A$\cap$B $\in\hat{\delta}_s$LC**

Proof. (i) Since A, B $\in\hat{\delta}_s$LC*, there exist $\hat{\delta}_s$-open sets U, V and closed sets F, G such that A = U$\cap$F and B = V $\cap$ G. Now, A$\cap$B = (U$\cap$F) $\cap$ (V$\cap$G) = (U$\cap$V) $\cap$(F$\cap$G) $\in\hat{\delta}_s$LC*.
The proof of (ii) and (iii) are similar to the proof of (i).

Definition 3.22. A subset A of an ideal topological space (X, τ, I) is called $\hat{\delta}_s$-locally δ-I-closed set if A = U$\cap$F where U is $\hat{\delta}_s$-open and F is δ-I-closed.

The class of all $\hat{\delta}_s$-locally δ-I-closed set is denoted by $\hat{\delta}_s\delta_I$LC (X, τ, I) or simply $\hat{\delta}_s\delta_I$LC.

Definition 3.23. For a subset A of an ideal space (X, τ, I), A$\in\hat{\delta}_s\delta_I$LC* if A = U$\cap$F where U is δ-I-open and F is $\hat{\delta}_s$-closed.

Theorem 3.24. Let A be a subset of an ideal space (X, τ, I). Then the following holds

(a) If A$\in\hat{\delta}_s\delta_I$LC then A$\in\hat{\delta}_sLC, A\in\hat{\delta}_s$LC*, A$\in$GLC, A$\in$GLC*

(b) If A$\in\hat{\delta}_s\delta_I$LC* then A$\in\hat{\delta}_sLC, A\in\hat{\delta}_s$LC**, A$\in$GLC, A$\in$GLC**.

Proof. The proof follows from the Definitions and Remark 2.14

The following examples shows that the converse is not hold always.

Example 3.25. Let X = {a, b, c, d}, τ = {X, ϕ, {a}, {a, d}} and I = {ϕ, {a}}. Let A = {b, c}. Then A$\in\hat{\delta}_s$LC*, GLC, GLC* but not in $\hat{\delta}_s\delta_I$LC. Let B = {a, d}. Then B$\in\hat{\delta}_s$LC**, GLC, GLC** but not in $\hat{\delta}_s\delta_I$LC.

Example 3.26. Let X = {a, b, c, d}, τ = {X, ϕ, {a, b, c}} and I = {ϕ, {a}}. Let A = {a, b, d}. Then A$\in\hat{\delta}_s$LC but not in $\hat{\delta}_s\delta_I$LC. Let B = {a, b}. Then B$\in\hat{\delta}_s$LC but not in $\hat{\delta}_s\delta_I$LC*.

Theorem 3.27. For a subset A of (X, τ, I), the following are equivalent.

(i) $A \in \hat{\delta}_s\delta_I LC(X, \tau, I)$.
(ii) $A = U \cap \sigma cl(A)$ for some $\hat{\delta}_s$ - open set U.
(iii) $\sigma cl(A) - A$ is $\hat{\delta}_s$ - closed.
(iv) $A \cup (X - \sigma cl(A))$ is $\hat{\delta}_s$ - open.

Proof. (i)⇒(ii) If $A \in \hat{\delta}_s\delta_I LC$, then there exist a $\hat{\delta}_s$ – open set U and a δ-I-closed set F such that $A = U \cap F$. Clearly $A \subset U \cap \sigma cl(A)$. Since F is δ-I-closed, $\sigma cl(A) \subset \sigma cl(F) = F$ and so $U \cap \sigma cl(A) \subset U \cap F = A$. Therefore $A = U \cap \sigma cl(A)$ for some $\hat{\delta}_s$ - open set U.
(ii)⇒(i) Since U is $\hat{\delta}_s$ – open and $\sigma cl(A)$ is δ-I-closed, we have $A = U \cap \sigma cl(A) \in \hat{\delta}_s\delta_I LC$.
(iii)⇒(iv) Let $F = \sigma cl(A) - A$. By assumption F is $\hat{\delta}_s$-closed and $X - F = X - (\sigma cl(A) - A) = A \cup (X - \sigma cl(A))$. Since $X - F$ is $\hat{\delta}_s$-open, we have $A \cup (X - \sigma cl(A))$ is $\hat{\delta}_s$-open.
(iv)⇒(iii) Let $U = A \cup (X - \sigma cl(A))$. Then U is $\hat{\delta}_s$-open, by hypothesis. This implies that $X - U$ is $\hat{\delta}_s$-closed and $X - U = X - (A \cup (X - \sigma cl(A)) = \sigma cl(A) \cap (X - A) = \sigma cl(A) - A$. Thus $\sigma cl(A) - A$ is $\hat{\delta}_s$-closed.
(ii)⇒(iv) Let $A = U \cap \sigma cl(A)$ for some $\hat{\delta}_s$-open set U. Now $A \cup (X - \sigma l(A)) = (U \cap \sigma cl(A)) \cup (X - \sigma cl(A)) = (U \cup (X - \sigma cl(A))) \cap (\sigma cl(A) \cup (X - \sigma cl(A)) = (U \cup (X - \sigma cl(A))) \cap X = U \cup (X - \sigma cl(A))$ is $\hat{\delta}_s$-open.
(iv)⇒(ii) Let $U = A \cup (X - \sigma cl(A))$. Then U is $\hat{\delta}_s$-open. Now $U \cap \sigma cl(A) = (A \cup (X - \sigma cl(A))) \cap \sigma cl(A) = (\sigma cl(A) \cap A) \cup (\sigma cl(A) \cap X - \sigma cl(A)) = A \cup \phi = A$. Therefore $A = U \cap \sigma cl(A)$ for some $\hat{\delta}_s$-open set U.

Theorem 3.28. Let (X, τ, I) be an ideal space and A be a subset of X. If $A \in \hat{\delta}_s\delta_I LC$ and $\sigma cl(A) = X$, then A is $\hat{\delta}_s$-open.

Proof. If $A \in \hat{\delta}_s\delta_I LC$, then by Theorem 3.27, $A \cup (X - \sigma cl(A))$ is $\hat{\delta}_s$-open. Since $\sigma cl(A) = X$, then A is $\hat{\delta}_s$-open.

Theorem 3.29. Let A and B be subsets of an ideal space (X, τ, I). Then the following holds.
(i) If $A, B \in \hat{\delta}_s\delta_I LC$ then $A \cap B \in \hat{\delta}_s\delta_I LC$
(ii) If $A, B \in \hat{\delta}_s\delta_I LC^*$, then $A \cap B \in \hat{\delta}_s\delta_I LC^*$.

Proof. (i) It follows from Definition 3.22 and Theorem 3.27(ii) there exist a $\hat{\delta}_s$-open sets U and V such that $A = U \cap \sigma cl(A)$ and $B = V \cap \sigma cl(B)$. Then $A \cap B = (U \cap \sigma cl(A)) \cap (V \cap \sigma cl(B)) = (U \cap V) \cap (\sigma cl(A) \cap \sigma cl(B))$. Since $U \cap V$ is $\hat{\delta}_s$-open and $\sigma cl(A) \cap \sigma cl(B)$ is δ-I-closed, $A \cap B \in \hat{\delta}_s\delta_I LC$.
(ii) Form the Definition 3.23 there exist δ-I-open sets U and V and $\hat{\delta}_s$-closed sets, F and G such that $A = U \cap F$ and $B = V \cap G$. Now, $A \cap B = (U \cap F) \cap (V \cap G) = (U \cap V) \cap (F \cap G) \in \hat{\delta}_s\delta_I LC^*$, since by Theorem 4.23[9] $F \cap G$ is $\hat{\delta}_s$-closed and $U \cap F$ is δ-I-closed.

Theorem 3.30. Let A and B be subsets of (X, τ, I). Then the following holds.
(i) If $A \in \hat{\delta}_s\delta_I LC$ and B is δ-I-closed, then $A \cap B \in \hat{\delta}_s\delta_I LC$
(ii) If $A \in \hat{\delta}_s\delta_I LC^*$ and B is either δ-I-open or δ-I-closed, then $A \cap B \in \hat{\delta}_s\delta_I LC$.

Proof. (i) If $A \in \hat{\delta}_s\delta_I LC$, then there exist a $\hat{\delta}_s$-open set U and a δ-I-closed set F in (X, τ, I), such that $A = U \cap F$. Now, $A \cap B = (U \cap F) \cap B = U \cap (F \cap B) \in \hat{\delta}_s\delta_I LC$.
(ii) If $A \in \hat{\delta}_s\delta_I LC^*$, then there exists δ-I-open set U and $\hat{\delta}_s$-closed set F such that $A = U \cap F$. Now, $A \cap B = (U \cap F) \cap B = (U \cap B) \cap F \in \hat{\delta}_s\delta_I LC^*$, for B is δ-I-open. For B is δ-I-closed, by Lemma 2.12, $F \cap B$ is $\hat{\delta}_s$-closed and so $A \cap B \in \hat{\delta}_s\delta_I LC^*$.

IV. $\hat{\delta}_s$ - NORMAL SPACES

In this section we introduce and study a class of normal space known as $\hat{\delta}_s$-normal spaces in an ideal topological spaces.

Definition 4.1. [6] A space (x, τ) is said to be g-normal if for every disjoint g-closed sets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.
Definition 4.2. [8] An ideal space (X, τ, I) is said to be I_g-normal space if for every pair of disjoint closed sets A and B, there exist disjoint I_g-open sets U and V such that $A \subset U$ and $B \subset V$.
Definition 4.3. An ideal space (X, τ, I) is said to be $\hat{\delta}_s$-normal space if for every pair of disjoint closed sets A and B, there exist disjoint $\hat{\delta}_s$-open sets U and V such that $A \subset U$ and $B \subset V$.

Since every $\hat{\delta}_s$-open set is I_g-open, every $\hat{\delta}_s$- normal space is I_g-normal. The following example shows that the converse is fails in some cases.

Example 4.4. Let $X=\{a, b, c, d\}$, $\tau = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}$ and $I=\{\phi, \{b\}, \{d\}, \{b, d\}\}$. The (X, τ, I) is I_g-normal but not $\hat{\delta}_s$-normal.

Theorem 4.5. Let (X, τ, I) be an ideal space. Then the following are equivalent:

i) X is $\hat{\delta}_s$-normal.
ii) For every pair of disjoint closed sets A and B, there exist disjoint $\hat{\delta}_s$-open sets U and V such that $A \subset U$ and $B \subset V$.
iii) For every closed set A and open set V containing A, there exists a $\hat{\delta}$ -open set U such that $A \subset U \subset \sigma cl(U) \subset V$.
iv) For any disjoint closed sets A and B, there exist a $\hat{\delta}_s$-open set U such that $A \subset U$ and $\sigma cl(U) \cap B = \phi$.
v) For each pair of disjoint closed sets A and B in (X, τ, I), there exist $\hat{\delta}_s$-open sets, U and V such that $A \subseteq U$, $B \subseteq V$ and $\sigma cl(U) \cap cl(V) = \phi$.

Proof. (i)$\Rightarrow$(ii). The proof is follows from the definition of $\hat{\delta}_s$-normal space.
(ii)$\Rightarrow$(iii). Let A be a closed set and V be an open set containing A. Then $X - V$ is the closed set distinct from A, therefore there exist disjoint $\hat{\delta}_s$-open sets U and W such that $A \subset U$ and $X - V \subset W$. since $U \cap W = \phi$, $U \subset X - W$. Therefore $U \subset X - W \subset V$ and since $X - W$ is $\hat{\delta}_s$-closed, then $A \subset U \subset \sigma cl(U) \subset \sigma cl(X - W) \subset V$, since V is open and hence semi-open.
(iii) $\Rightarrow$(iv). Let A and B be any disjoint closed sets. Then $X - B$ is an open set such that $A \subset X - B$ and by hypothesis there exist a $\hat{\delta}_s$-open set U such that $A \subset U \subset \sigma cl(U) \subset X - B \subset V$. $\sigma cl(U) \subset X - B$. Hence $\sigma cl(U) \cap B = \phi$.
(iv)$\Rightarrow$(v). Let A and B are closed sets in (X, τ, I). Then by assumption, there exists $\hat{\delta}_s$-open set U containing A such that $\sigma cl(U) \cap B = \phi$. Since $\sigma cl(U)$ is closed, $\sigma cl(U)$ and B are distinct closed set in (X, τ, I). Therefore again by assumption, there exist a $\hat{\delta}_s$-open set V containing B, $\sigma cl(V) \cap \sigma cl(U) = \phi$. Hence $\sigma cl(U) \cap \sigma cl(V) = \phi$.
(v)$\Rightarrow$(i) Let A and B be any disjoint closed sets of (X, τ, I). By assumption, there exist $\hat{\delta}_s$-open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $\sigma cl(U) \cap \sigma cl(V) = \phi$. We have $U \cap V = \phi$ and thus (X, τ, I) is $\hat{\delta}_s$-normal.

Theorem 4.6. Let (X, τ, I) be a $\hat{\delta}_s$-normal space. If F is closed and A is a $\hat{\delta}_s$-closed set such that $A \cap F = \phi$, then there exist disjoint $\hat{\delta}_s$-open set U and V such that $A \subset U$ and $F \subset V$.

Proof. Since $A \cap F = \phi$, $A \subset X - F$ Where $X - F$ is open and hence semi-open, By hypothesis $\sigma cl(A) \subset X - F$. Since $\sigma cl(A) \cap F = \phi$ and X is $\hat{\delta}_s$-normal and $\sigma cl(A)$ is closed, there exist disjoint $\hat{\delta}_s$-open sets U and V such that $\sigma cl(A) \subset U$ and $F \subset V$.

Corollary 4.7. Let (X, τ, I) be a $\hat{\delta}_s$-normal space. If F is of δ-I-closed and A is $\hat{\delta}_s$-closed such that $A \cap F = \phi$, then there exists disjoint $\hat{\delta}_s$-open set U and V such that $A \subset U$ and $F \subset V$.

Proof. The Poof follows from the fact that every δ-I-closed set is closed.

Corollary 4.8. Let (X, τ, I) be a $\hat{\delta}_s$-normal space. If F is δ-closed and A is $\hat{\delta}_s$-closed such that $A \cap F = \phi$, then there exist disjoint $\hat{\delta}_s$-open set U and V such that $A \subset U$ and $F \subset V$.

Proof. The proof follows from the fact that every δ-closed set is δ-I-closed.

Corollary 4.9. Let (X, τ, I) be $\hat{\delta}_s$-normal space. If F is regular closed and A is $\hat{\delta}_s$-closed set such that $A \cap F = \phi$, then there exists disjoint $\hat{\delta}_s$-open sets U and V such that $cl(A) \subset U$ and $F \subset V$.

Definition 4.10. An ideal space (X, τ, I) is said to be $\hat{\delta}_s$-I-normal if for each pair of disjoint $\hat{\delta}_s$-closed sets A and B, there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

Theorem 4.11. Let (X, τ, I) be an ideal space. Then the following are equivalent.

(i) $\hat{\delta}_s$-I-normal.

(ii) For each pair of disjoint $\hat{\delta}_s$-closed sets A and B, there exist disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

(iii) For every $\hat{\delta}_s$-closed set A and every $\hat{\delta}_s$-open set V containing A, there exist an open set U of X such that $A \subset U \subset cl(U) \subset V$.

(iv) For each disjoint pair of $\hat{\delta}_s$-closed sets A and B, there exist an open set U such that $A \subset U$ and $cl(U) \cap B = \phi$.

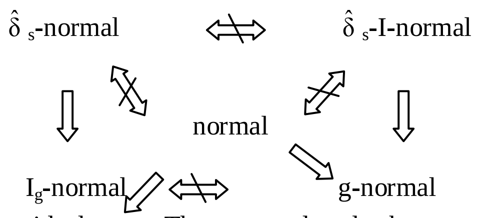
Proof. (i)$\Rightarrow$(ii). The proof follows from the definition.

(ii)$\Rightarrow$(iii). Let A be a $\hat{\delta}_s$-closed set and V be a $\hat{\delta}_s$-open set containing A. Then X–V is $\hat{\delta}_s$-closed. Hence A and X–V are disjoint $\hat{\delta}_s$-closed sets. By hypothesis there exists disjoint open sets U and W such that $A \subset U$ and $X–V \subset W$. Since $U \cap W = \phi$, $U \subset X–W$ or $W \subset X–U$. Therefore $U \subset X–W \subset V$. Therefore $A \subset U \subset cl(U) \subset cl(X–W) = X–W \subset V$.

(iii) $\Rightarrow$(iv). Let A and B are disjoint $\hat{\delta}_s$-closed sets. Then X–B is a $\hat{\delta}_s$-open set containing A and therefore by hypothesis, there exist an open set U such that $A \subset U \subset cl(U) \subset X–B$. Therefore $cl(U) \cap B = \phi$.

(iv) $\Rightarrow$(i). Let A and B are disjoint $\hat{\delta}_s$-closed sets. By hypothesis there exists an open set U containing A and $cl(U) \cap B = \phi$. If we take V=X–cl(U), then V is an open set containing B. Therefore U and V are disjoint open sets such that $A \subset U$ and $B \subset V$. This proves (X, τ, I) is $\hat{\delta}_s$-I-normal.

Remark 4.12. The following implications hold for an ideal space (X, τ, I). Here A $\Rightarrow$ B means A implies B, but not conversely and A $\not\Leftrightarrow$ B means the implications not hold on either side.



Theorem 4.13. Let (X, τ, I) be an ideal space. Then every closed subspace of a $\hat{\delta}_s$-normal space is $\hat{\delta}_s$-normal.

Proof. Let G be a closed subspace of a $\hat{\delta}_s$-normal space (X,τ, I). Let τ_1 be the relative topology for G. Let E_1 and F_1 be any two disjoint τ_1-closed subsets of G. Then there exist τ-closed subsets E and F such that $E_1 = G \cap E$ and $F_1 = G \cap F$. Since G and E are τ–closed, E_1 is also τ-closed and F_1 is also τ-closed. Thus E_1 and F_1 are disjoint subsets of $\hat{\delta}_s$-normal space (X, τ, I). Therefore, there exist disjoint $\hat{\delta}_s$-open sets U and V such that $E_1 \subset U$ and $F_1 \subset V$. Hence for every disjoint closed sets E_1 and F_1 in G, we can find disjoint $\hat{\delta}_s$-open sets U and V such that $E_1 \subset U$ and $F_1 \subset V$. Therefore $E_1 \subset U \cap G$ and $F_1 \subset V \cap G$, where $U \cap G$ and $V \cap G$ are disjoint $\hat{\delta}_s$-open sets in G. Hence (G, τ, I) is $\hat{\delta}_s$-normal.

V. $\hat{\delta}_S$-CONNECTED SPACE

In this section we define and study a connected space known as $\hat{\delta}_s$-connected space.

Definition 5.1. X = $A \cup B$ is said to be a $\hat{\delta}_s$-separation of X if A and B are non-empty disjoint $\hat{\delta}_s$-open sets. If there is no $\hat{\delta}_s$-separation of X, then X is said to be $\hat{\delta}_s$-connected. Otherwise it is said to be $\hat{\delta}_s$-disconnected.

Note 5.2. If X=A∪B is a $\hat{\delta}_s$-separation then A = X–B and B = X–A. Hence A and B are $\hat{\delta}_s$-closed.

Theorem 5.3. An ideal space is (X, τ, I) is $\hat{\delta}_s$-connected if and only if the only subsets which are both $\hat{\delta}_s$-open and $\hat{\delta}_s$-closed are X and ϕ.

Proof. Necessity - Let (X, τ, I) be a $\hat{\delta}_s$-connected space. Suppose that A is a proper subset which is both $\hat{\delta}_s$-open and $\hat{\delta}_s$-closed then X=A∪(X–A) is a $\hat{\delta}_s$-separation of X. Which is a contradiction.

Sufficiency - Let ϕ be the only subset which is both $\hat{\delta}_s$-open and $\hat{\delta}_s$-closed. Suppose X is not $\hat{\delta}_s$-connected, then X=A∪B where A and B are non-empty disjoint $\hat{\delta}_s$-open subsets which is contradiction.

Definition 5.4. Let Y be a subset of X. Then Y=A∪B is said to be $\hat{\delta}_s$-separation of Y if A and B are non-empty disjoint $\hat{\delta}_s$-open sets in X. If there is no $\hat{\delta}_s$-separation of Y then Y is said to be $\hat{\delta}_s$-connected subset of X.

Theorem 5.5. Let (X, τ, I) be an ideal topological space. If X is $\hat{\delta}_s$-connected, then X cannot be written as the union of two disjoint non-empty $\hat{\delta}_s$-closed sets.

Proof. Suppose not, that is X = A∪B, where A and B are $\hat{\delta}_s$-closed sets, A≠ϕ, B≠ϕ and A∩B≠ϕ, Then A = X–B and B = X–A. Since A and B are $\hat{\delta}_s$-closed sets which implies that A and B are $\hat{\delta}_s$-open sets. Therefore X is not $\hat{\delta}_s$-connected. Which is a contradiction.

Corollary 5.6. Let (X, τ, I) be an ideal topological space. If X is $\hat{\delta}_s$-connected, then X cannot be written as the union of two disjoint non-empty δ-closed sets.

Corollary 5.7. Let (X, τ, I) be an ideal topological space. If X is $\hat{\delta}_s$-connected, then X cannot be written as the union of two disjoint non-empty δ-I-closed sets.

Definition 5.8. Two non-empty subsets A and B of an ideal space (X, τ, I) are called $\hat{\delta}_s$ - separated if A∩σcl(B) = σcl(A)∩B = ϕ.

Remark 5.9. Since cl(A)⊂σcl(A), A∩cl(B) = cl(A)∩B⊂A∩σcl(B)= σcl(A)∩B=ϕ. Here $\hat{\delta}_s$ - separated sets are separated. But the converse need not be true as shown in the following example.

Example 5.10. Let X={a,b,c,d}, τ={X,ϕ,{b},{a,b},{b,c},{a,b,c},{a,b,d}} and I={ϕ, {c},{d},{c,d}}. Let A={c},B={d}. Then A∩cl(B)=cl(A)∩B={c}∩{d}=ϕ. But A∩σcl(B)={c}∩X={c}≠ϕ and σcl(A)∩B= X∩{d}≠ϕ. Therefore A and B are separated but not $\hat{\delta}_s$-separated.

Theorem 5.11. Let (X, τ, I) be an ideal space. If A is $\hat{\delta}_s$ - connected set of X and H, G are $\hat{\delta}_s$ - separated sets of X with A⊂H∪G, then either A⊂H or A⊂G.

Proof. Let A⊂H∪G, Since A = (A∩H)∪(A∩G), then (A∩G)∩σcl(A∩H)⊂ G∩σcl(H) = ϕ. Similarly, we have σcl(A∩G)∩(A∩H) = ϕ. Suppose that, A∩H and A∩G are non-empty, then A is not $\hat{\delta}_s$ – connected. This is a contradiction. Thus either A∩H = ϕ or A∩G = ϕ. Which implies that A⊂H or A⊂G.

Theorem 5.12. If A is $\hat{\delta}_s$-connected set of an ideal topological space (X, τ, I) and A⊂B⊂σcl(A), then B is $\hat{\delta}_s$ - connected.

Proof. Suppose that B is not $\hat{\delta}_s$ - connected. There exist $\hat{\delta}_s$ - separated sets H and G such that B = H∪G. This implies that H and G are non-empty and G∩σcl(H) = H∩σcl(G) = ϕ. By Theorem 5.11, we have either A⊂H or A⊂G. Suppose A⊂G. Then σcl(A)⊂ σcl(G) and H∩σcl(A) = ϕ. This implies that H⊂B⊂σcl(A) and H = σcl(A)∩H = ϕ. Thus H is an empty set. Since H is non-empty, there is a contradiction. Similarly, suppose A⊂H, then G is empty. Therefore contradiction. Hence B is $\hat{\delta}_s$ - connected.

Corollary 5.13. If A is a $\hat{\delta}_s$-connected set in an ideal space (X, τ, I), then σcl(A) is $\hat{\delta}_s$-connected.

Proof. The proof is obvious.

Corollary 5.14. If A is a $\hat{\delta}_s$-connected set in an ideal space (X, τ, I), then cl(A) is $\hat{\delta}_s$-connected.

Corollary 5.15. If A is a $\hat{\delta}_s$-connected set in an ideal space (X, τ, I), then cl*(A) is $\hat{\delta}_s$-connected.

Corollary 5.16. If A is a $\hat{\delta}_s$-connected set in an ideal space (X, τ, I), then A* is $\hat{\delta}_s$-connected.

Proof. The proof is obvious.

Theorem 5.17. If $\{A_\alpha : \alpha\in\Delta\}$ is a non-empty family of $\hat{\delta}_s$ - connected sets of an ideal space (X, τ, I) with $\bigcap_{\alpha\in\Delta} A_\alpha \neq \phi$, then $\bigcup_{\alpha\in\Delta} A_\alpha$ is $\hat{\delta}_s$ - connected.

Proof. Suppose $\bigcup_{\alpha\in\Delta} A_\alpha$ is not $\hat{\delta}_s$-connected. Then we have $\bigcup_{\alpha\in\Delta} A_\alpha = H\cup G$, where H and G are $\hat{\delta}_s$-separated sets in X. Since $\bigcap_{\alpha\in\Delta} A_\alpha \neq \phi$, $x\in \bigcap_{\alpha\in\Delta} A_\alpha$. Also since $x\in \bigcap_{\alpha\in\Delta} A_\alpha$, either $x\in H$ or $x\in G$. Suppose $x\in H$. Since $x\in A_\alpha$ for each $\alpha\in\Delta$, A_α and H intersects for each α. By Theorem 5.11, $A_\alpha\subset H$ or $A_\alpha\subset G$. Since H and G are disjoint $A_\alpha\subset H$ for all $\alpha\in\Delta$ and hence $\bigcup_{\alpha\in\Delta} A_\alpha\subset H$. Which implies that G is empty. This is a contradiction. Similarly, suppose $x\in G$. then we have H is empty. This is a contradiction. Thus $\bigcup_{\alpha\in\Delta} A_\alpha$ is $\hat{\delta}_s$-connected.

REFERENCES

[1]. Balachandran.K, P.Sundaram and H. Maki, Generalized locally closed sets and glc – continuous functions, Indian J.Pure Appl. Math. 27(3) (1996), 235-244.

[2]. Bourbaki.N, General Topology, Part I, Addison – Wesley, Reading, Mass, 1966.

[3]. Dontchev.J, M.Ganster and T.Noiri, Unified approach of generalized closed sets via topological ideals, Math. Japonica, 49(1999), 395-401.

[4]. Kuratowski.K, Topology, Vol.I, Academic press, New York, 1966.

[5]. Levine.N, Generalized closed sets in topology, Rend. Circ. Mat. Palermo, 19(2), (1970), 89-96.

[6]. Munshi, B.M., Separation Axioms, Acta Ciencia Indica, 12(1986), 140-144.

[7]. Navaneethakrishnan.M and D.Sivaraj, Generalized locally closed sets in Ideal topological spaces, Bulletin of the Allahabad Mathematical Society, Golden Jubilee year volume 2008.

[8]. Navaneethakrishnan, M. and J. Paulraj Joseph and D. Sivaraj, I_g-normal and I_g-regular spaces. Acta. Math. Hungar, 2009.

[9]. Navaneethakrishnan.M, P.Periya Samy, S.Pious Missier, Between δ-I-closed sets and g-closed sets in ideal topological spaces. Int. J. Mod. Eng. Res. (2015), 39-45.

[10]. Vaidyanathaswamy.R, Set topology, Chelsea Publishing Company, 1946.

[11]. Yuksel.S, A.Acikgoz and T.Noiri, On δ-I-continuous functions, Turk J.Math, 29(2005), 39-51.