OPEN ACCESS

International Journal Of Modern Engineering Research (IJMER)

Biorthogonal Nonuniform Multiwavelet Packets associated with nonuniform multiresolution analysis with multiplicity D

[1]Nadya A. S. Atlouba, [2]Ajit Paul, [3]Naser R. Naser
[1], [2], [3]Department of Mathematics and Statistics,
SHIATS, (Deemed to be University), Allahabad, India-211 007

ABSTRACT:- A biorthogonal wavelet packets associated with nonuniform multiresoltion analysis (NUMRA) was introduced by Firdous shah, In this paper we generalize and define the biorthogonal nonuniform multiwavelet packets associated by nonuniform multiresolution analysis with multiplicity D (NUMRA-D). Further from the meaning of Fourier transform we study their characteristics.

KEYWORDS:- NUMRA with multiplicity D, nonuniform Multiwavelet, biorthogonal wavelet packets, Riesz basis, Fourier transform.

I. INTRODUCTION

In [9,10] Gabardo and Nashed considered a generalization of Mallat's[16] theory of MRA based on spectral pairs where the translation set is of the form $\{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N - 1$, r is an odd integer and r, N are relatively prime, which is called NUMRA. In [14] we provided the necessary and sufficient condition for the existence of NUMRAMW multiplicity D. G.Gripenberg and X. Wang gave the characterization theorem for dyadic orthonormal wavelet in $L^2(\mathbb{R})$ Wavelet packets constructed by coifman [15] it was gave rise to a large class of orthonormal bases of $L^2(\mathbb{R})$ into direct sum of its closed subspaces Wavelet packets. Behera[2,3] has constructed nonuniform wavelet packets associated with NUMRA. In [1] we constructed multiwavelet packets associated with nonuniform multiresolution analysis with multiplicity D (NUMRA-D). Firdous [8] introduce the notion of biorthogonal wavelet packets associated with nonuniform multiresolution analysis. Our result extends the biorthogonal wavelet packets of Firdous shah into NUMRA-D in has constructed nonuniform wavelet packets associated with NUMRA.

II. BASIC DEFINITIONS AND NOTATION

We will state some important preliminaries and notation in this section that we are need it in the recent paper. First we recall the definition of $NUMRA - D$ as defined in [14], some of its properties and the associate multiwavelet packets [1-3, 8, 14] as follows:

Definition 2.1. A nonuniform multiresolution analysis with multiplicity D for dilation $\theta \equiv 2N\alpha$ and translation Λ, is a collection $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following axioms:

$(P1)$ $V_j \subset V_{j+1},\ for\ all\ j \in \mathbb{Z},$

$(P2)$ $f(\cdot) \in V_j \quad if\ and\ only\ if\ f(\theta\ \cdot) \in V_{j+1},\ for\ all\ j \in \mathbb{Z},$

$(P3)$ $\bigcap_{j\in\mathbb{Z}} V_j = \{0\},$

$(P4)$ $\bigcup_{j\in\mathbb{Z}} V_j$ $is\ dense\ in\ L^2(\mathbb{R})$, and

(P5) There exist functions $\varphi^1, \varphi^2, \dots, \varphi^D \in V_0$, called the scaling functions, such that the collection $\{\varphi^d(\cdot - \lambda) : \lambda \in \Lambda, 1 \leq d \leq D\}$ is a complete orthonormal basis for V_0.

In the axiom $(P5)$, the set of scaling functions $\Phi \equiv \{\varphi^1, \varphi^2, \dots, \varphi^D\}$ is called multiscaling function of multiplicity D. When $N \geq 1$, the dilation factor of θ ensures that $\theta\Lambda \subset 2\mathbb{Z} \subset \Lambda$.

In the theory of multresolution analysis, another sequence $\{W_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ is defined by $W_j = V_{j+1} \ominus V_j,\ j \in \mathbb{Z}$ and $\ominus$ denotes the orthogonal complement of V_j in V_{j+1}, for an $NUMRA - D$ $\{V_j\}_{j\in\mathbb{Z}}$ with dilation factor θ. These subspaces hold the scaling property of $\{V_j\}_{j\in\mathbb{Z}}$, and we have :

$$L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j = V_0 \oplus (\oplus_{j\geq 0} W_j) \qquad (1)$$

A set of functions $\{\psi^l : 1 \leq l \leq (\theta - 1)D\} := \Psi$ in $L^2(\mathbb{R})$ is said to be a nonuniform multiwavelet associated with the $NUMRA - D$ $\{V_j\}_{j\in\mathbb{Z}}$ if the collection $\{\psi^l(\cdot - \lambda) : 1 \leq l \leq (\theta - 1)D,\ \lambda \in \Lambda\}$ forms an orthonormal basis for W_0. We call Ψ to be an $NUMRA - D$ multiwavelet. In view of properties of W_j, the collection

$$\{\theta^{j/2}\, \psi^l(\theta^j \ \cdot - \lambda) : j \in \mathbb{Z}, 1 \leq l \leq (\theta - 1)D, \qquad \lambda \in \Lambda\}$$

forms an orthonormal basis for $L^2(\mathbb{R})$ if Ψ is a nonuniform multiwavelet.

Below we mention a result obtained in [14] that will be used in sequel.

Theorem 2.2. Suppose $\{V_j\}_{j\in\mathbb{Z}}$ is an $NUMRA - D$ with dilation θ and translation Λ. If there exist L functions $\psi^k, 1 \leq k \leq L$, in V_1 such that the family of functions

$$V_1' := \{\varphi^d(\cdot - \lambda), \qquad \psi^k(\cdot - \lambda) : 1 \leq d \leq D;\ 1 \leq k \leq L;\ \lambda \in \Lambda\}$$

forms an orthonormal system for the generating subspace V_1, then; $L = (\theta - 1)D$ is a necessary and sufficient condition such that the above system is complete in V_1.

Proposition 2.3. Suppose $\{V_j\}_{j\in\mathbb{Z}}$ is an $NUMRA - D$ with dilation θ and translation Λ. Then the space V_1 consists precisely of the functions $f \in L^2(\mathbb{R})$ whose Fourier transform can be written as

$$\hat{f}(\theta\xi) = \sum_{d=1}^{D} m^{f,d}(\xi)\, \widehat{\varphi^d}(\xi), \qquad a.e. \quad \xi \in \mathbb{R}, \tag{2}$$

for locally L^2 functions $m^{f,d}$ for $1 \leq d \leq D$ together with

$$m^{f,d}(\xi) = m_1^{f,d}(\xi) + e^{-\frac{2\pi i\xi r}{N}} m_2^{f,d}(\xi), \tag{3}$$

where $m_1^{f,d}$ and $m_2^{f,d}$, $d = 1, 2, \ldots, D$, are locally L^2, $1/2$-periodic functions.

In the above proposition the space V_1 consist functions $f \in L^2(\mathbb{R})$. Since $(1/\theta)f(x/\theta) \in V_0$, there exists a sequence $\{a_\lambda^f = (a_\lambda^{f,1}, a_\lambda^{f,2}, \ldots, a_\lambda^{f,D})\}_{\lambda\in\Lambda}$ satisfying $\sum_{d=1}^{D}\sum_{\lambda\in\Lambda}\left|a_\lambda^{f,d}\right|^2 < \infty$ such that

$$\frac{1}{\theta} f\left(\frac{x}{\theta}\right) = \sum_{d=1}^{D}\sum_{\lambda\in\Lambda} a_\lambda^{f,d}\, \varphi^d(x - \lambda), \tag{4}$$

or, equivalently, by taking the Fourier transform of both sides of the previous equation, we obtained the result (2.1) and (2.2), with

$$m^{f,d}(\xi) = \textstyle\sum_{\lambda\in\Lambda} a_\lambda^{f,d} e^{-2\pi i\xi\lambda}, \tag{5}$$

$$m_1^{f,d}(\xi) = \sum_{m\in\mathbb{Z}} a_{2m}^{f,d}\ e^{-4\pi i\xi m}, \ and \ \ m_2^{f,d}(\xi) = \sum_{m\in\mathbb{Z}} a_{2m+\frac{r}{N}}^{f,d}\ e^{-4\pi i\xi m}. \tag{6}$$

We denote by $\widehat{\Phi}$ and $\widehat{\Psi}$ column vectors in C^D and in C^L as $\widehat{\Phi} = \{\widehat{\varphi^1}, \ldots, \widehat{\varphi^D}\}$ and $\widehat{\Psi} = \{\widehat{\psi^1}, \ldots, \widehat{\psi^L}\}$, respectively. In particular, since $\varphi^d(x) \in V_0 \subset V_1$, from Proposition 2.3 there are locally L^2 functions $m_0^{d,d'}$, for $1 \leq d, d' \leq D$, such that, for a.e.

$$\widehat{\varphi^d}(\theta\xi) = \sum_{d'=1}^{D} m_0^{d,d'}(\xi)\, \varphi^{d'}(\xi) = \sum_{d'=1}^{D}\left(m_{01}^{d,d'}(\xi) + e^{-\frac{2\pi i\xi r}{N}} m_{02}^{d,d'}(\xi)\right)\varphi^{d'}(\xi). \tag{7}$$

Taking $m_{01}^{d,d'} \equiv m_{01}^{\varphi^d,d'}$ and $m_{02}^{d,d'} \equiv m_{02}^{\varphi^d,d'}$, in the matrix notation, this can be written as follows:

$$\widehat{\Phi}(\theta\xi) = M_0(\xi)\, \widehat{\Phi}(\xi), \ \ for\ a.e.\ \xi \in \mathbb{R},$$

with $M_0(\xi) = \left(M_{01}(\xi) + e^{-2\pi i\xi r/N} M_{02}(\xi)\right)$

, where the matrices M_{01} and M_{02} defined by

$M_{01} = \left(m_{01}^{d,d'}\right)_{1\leq d,d'\leq D}$ and $M_{02} = \left(m_{02}^{d,d'}\right)_{1\leq d,d'\leq D}$ are usually called low-pass filters (or scaling matrix filters) associated with the scaling family Φ.

Similarly, in case of $\psi^l(x) \in W_0 \subset V_1$, where $W_0 = \overline{span}\{\psi^l(x - \lambda) : 1 \leq l \leq L;\ \lambda \in \Lambda\}$, there are locally L^2 functions $m_1^{\psi^l,d}$, for $1 \leq d \leq D$ and $1 \leq l \leq L$ such that, for a.e. $\xi \in \mathbb{R}$,

$$\widehat{\psi^l}(\theta\xi) = \sum_{d=1}^{D} m_1^{\psi^l,d}(\xi)\widehat{\varphi^d}(\xi) = \sum_{d=1}^{D}\left(m_{11}^{\psi^l,d}(\xi) + e^{-2\pi i\xi r/N}\ m_{12}^{\psi^l,d}(\xi)\right)\varphi^d(\xi), \tag{8}$$

whose matrix notation is given as follows by considering $m_{11}^{l,d} \equiv m_{11}^{\psi^l,d}$ and $m_{12}^{l,d} \equiv m_{12}^{\psi^l,d}$:

$$\widehat{\Psi}(\theta\xi) = M_1(\xi)\,\widehat{\Phi}(\xi),\;\; for\ a.e.\;\; \xi \in \mathbb{R},$$

with $M_1(\xi) = M_{11}(\xi) + e^{-2\pi i\xi r/N} M_{12}(\xi)$, where the matrices M_{11} and M_{12} defined by

$$M_{11} = \left(m_{11}^{l,d}\right)_{1\le l\le L, 1\le d\le D}$$

and

$$M_{12} = \left(m_{12}^{l,d}\right)_{1\le l\le L, 1\le d\le D}$$

are usually called high-pass filters associated with the scaling family (Ψ, Φ).

Definition 2.4. The function $\omega_l^d : 1 \le l \le L,\;\; 1 \le d \le D$ as defined as follows $\widehat{\omega}_l^d(\xi) = \sum_{h=1}^{D} m^{d,l,h}\left(\frac{\xi}{\theta}\right)\widehat{\varphi}^h(\xi/\theta), a.e.\;\; \xi \in \mathbb{R}.$ (9)

will be called the basic nonuniform multiwavelet packets corresponding to the $NUMRA - D$ $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ associated with the dilation θ.

Note that (4.2) defines ω_l^d for every non-negative integer l and every d such that $1 \le d \le D$.

Now $w_0^d = \varphi^d,\; d = 1,..,D$ is a multiscaling function of multiplicity D while ω_l^d , $1 \le l \le \theta - 1,\; d = 1,\cdots, D$, are the basic multiwavelets associated with $NUMRA - D$. Now for any $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, we define $\omega_n^d : 1 \le d \le D$, recursively as follows. Suppose that $\{\omega_p^d : 1 \le d \le D,\; p \in \mathbb{N}_0\}$ are defined already. Then define $\omega_{q+\theta p}^d$, $0 \le q \le \theta - 1$, by

$$\omega_{q+\theta p}^d(x) = \sum_{h=1}^{D}\sum_{\lambda\in\Lambda}\theta\, a_\lambda^{d,q,h}\, \omega_p^h(\theta x - \lambda). \tag{10}$$

Note that (4.2) defines ω_n^d for $n \ge 0$. Taking Fourier transform we get

$$(\widehat{\omega}_{q+\theta p}^d)(\xi) = \sum_{h=1}^{D} m^{d,q,h}(\xi/\theta)\,\widehat{\omega}_p^h(\xi/\theta),\;\; 0 \le q \le \theta - 1,\;\; 1 \le d \le D. \tag{11}$$

Theorem 2.5.[1] Let $\{\omega_n^d : n \ge 0,\;\; 1 \le d \le D\}$ be the basic nonuniform multiwavelet packets associated with the $NUMRA - D$ $\{V_j\}$. Then

(i) $\{\omega_n^d(\cdot - \lambda) : (\theta)^j \le n \le (\theta)^{j+1} - 1,\; \lambda \in \Lambda,\; 1 \le d \le D\}$ is an orthonormal basis of W_j , $j \ge 0$.

(ii) $\{\omega_n^d(\cdot - \lambda) : 0 \le n \le (\theta)^j - 1,\; \lambda \in \Lambda, 1 \le d \le D\}$ is an orthonormal basis of V_j , $j \ge 0$.

(iii) $\{\omega_n^d(\cdot - \lambda) : n \ge 0,\; \lambda \in \Lambda,\; 1 \le d \le D\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Now as in (4), (7), (8), (9), (10) and (11) we can define:

$$\widehat{\widetilde{\varphi}}^d(\theta\xi) = \sum_{d'=1}^{D} \widetilde{m}_0^{d,d'}(\xi)\,\widetilde{\varphi}^{d'}(\xi) \tag{12}$$

$$\widetilde{\psi}^l(\theta\xi) = \sum_{d=1}^{D} \widetilde{m}_1^{\psi^l,d}(\xi)\,\widehat{\widetilde{\varphi}}^d(\xi) \tag{13}$$

$$\widehat{\widetilde{\omega}}_l^d(\xi) = \sum_{h=1}^{D} \widetilde{m}^{d,l,h}\left(\frac{\xi}{\theta}\right)\widehat{\widetilde{\varphi}}^h(\xi/\theta), a.e.\;\; \xi \in \mathbb{R} \tag{14}$$

$$\widetilde{\omega}_{q+\theta p}^d(x) = \sum_{h=1}^{D}\sum_{\lambda\in\Lambda}\theta\,\widetilde{a}_\lambda^{d,q,h}\,\widetilde{\omega}_p^h(\theta x - \lambda) \tag{15}$$

$$(\widehat{\widetilde{\omega}}_{q+\theta p}^d)(\xi) = \sum_{h=1}^{D}\widetilde{m}^{d,q,h}(\xi/\theta)\,\widehat{\widetilde{\omega}}_p^h(\xi/\theta), 0 \le q \le \theta - 1, 1 \le d \le D \tag{16}$$

And

$$m_i^{d,q,h} = \begin{cases} m_{0i}^{d,h} : for\ q = 0, 1 \le d, h \le D; \\ m_{0i}^{k,h} : for\ q \ne 0;\; k = q + (d-1)(\theta - 1); 1 \le d, h \le D; 1 \le q \le (\theta - 1); \end{cases} \quad i = 1,2$$

III. BIORTHOGONAL MULTIWAVELET PACKETS

Definition 3.1. A pair of function $f(x), \tilde{f}(x) \in L^2(\mathbb{R})$ are biorthogonal, if their translates satisfy

$$\langle f(.), \tilde{f}(. - \lambda)\rangle = \delta_{0,\lambda},\qquad \lambda \in \Lambda.$$

Where $\delta_{0,\lambda}$ is kronecker symbol.

If $\varphi^d(x), \tilde{\varphi}^d(x) \in L^2(\mathbb{R})$ are a pair of biorthogonal scaling functions, then

$$\langle \varphi^d(.), \tilde{\varphi}^d(. - \lambda)\rangle = \delta_{0,\lambda},\; \lambda \in \Lambda, 1 \le d \le D. \tag{17}$$

Moreover we say that $\psi_r^d(.),\; \tilde{\psi}_r^d(. - \lambda) \in L^2(\mathbb{R}), 1 \le r \le \theta - 1, 1 \le d \le D$ are pair of biorthogonal nonuniform multiwavelet associated with a pair of biorthogonal scaling functions $\varphi^d(.), \hat{\varphi}^d(. - \lambda)$ if the family $\{\psi_r^d(. - \lambda) : \lambda \in \Lambda, r = 1, \dots, \theta - 1, 1 \le d \le D\}$ is a riesz of w_0, and

$$\langle \varphi^d(.), \tilde{\psi}_r^d(.-\lambda)\rangle = 0\,,\ \lambda \in \Lambda, 1 \le r \le \theta - 1, 1 \le d \le D \qquad (18)$$
$$\langle \tilde{\varphi}^d(.), \psi_r^d(.-\lambda)\rangle = 0\,,\ \lambda \in \Lambda, 1 \le r \le \theta - 1, 1 \le d \le D \qquad (19)$$
$$\langle \psi_r^d(.), \tilde{\psi}_s^d(.-\lambda)\rangle = \delta_{r,s}\, \delta_{o,\lambda},\ \lambda \in \Lambda, 1 \le r, s \le \theta - 1, 1 \le d \le D \qquad (20)$$

Proposition 3.2. If $\psi_r^d(x), \tilde{\psi}_r^d(x) \in L^2(\mathbb{R}), 1 \le r \le \theta - 1$ are a pair of biorthogonal nonuniform $multiwavelet\ associated\ with\ a\ pair\ of\ biorthogonal\ scaling\ functions\ \varphi^d(x), \tilde{\varphi}^d(x),\ then\ L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \oplus \left(\bigoplus_{j\ge 0} W_j\right) = \bigoplus_{j\in\mathbb{Z}} \bigoplus_{r=1}^{\theta-1} W_j^r$
Where
$W_j^r = \overline{span}\,\{\psi_r^d((\theta)^j.-\lambda): \lambda \in \Lambda, j \in \mathbb{Z}, 1 \le r \le \theta - 1, d = 1, \dots, D\}$ (21)

Lemma 3.3. Let $\varphi^d(x), \tilde{\varphi}^d(x)$ be a pair of scaling functions. Then $\varphi^d(x), \tilde{\varphi}^d(x)$ are biorthogonal scaling if and only if $\sum_{d=1}^{D} \sum_{\lambda\in\Lambda} \varphi^d(\xi - \lambda)\, \overline{\hat{\tilde{\varphi}}^d(\xi - \lambda)} = 1$ a.e $\xi \in \mathbb{R}$.

Lemma 3.4. Asume that $\omega_q^d, \tilde{\omega}_q^d \in L^2(\mathbb{R}),\ 1 \le q \le \theta - 1, 1 \le d \le D$ are a pair of biorthogonal nonuniform multiwavelet associated with a pair of biorthogonal scaling functions $\omega_0^d(x)$ and $\tilde{\omega}_0^d(x)$. Then,

$$\sum_{\sigma=0}^{\theta-1} \sum_{h=1}^{D} m^{d,p,h}\left(\frac{\xi}{\theta} + 2\pi\sigma\right) \overline{\tilde{m}^{d,q,h}\left(\frac{\xi}{\theta} + 2\pi\sigma\right)} = \delta_{p,q},\ \ 0 \le p, q \le \theta - 1, 1 \le d \le D \qquad (22)$$

Proof. By (11), (14), (17), (21) and Lemma 3.3, we have

$$\begin{aligned}
\delta_{p,q} &= \sum_{\lambda\in\Lambda} \omega_p^d(\xi + 2\pi\lambda)\ \overline{\tilde{\omega}_q^d(\xi + 2\pi\lambda)} \\
&= \sum_{\lambda\in\Lambda} \sum_{h=1}^{D} m^{d,p,h}\big(\theta^{-1}(\xi + 2\pi\lambda)\big)\ \hat{\omega}_0^d\big(\theta^{-1}(\xi + 2\pi\lambda)\big) \\
&\quad \times \overline{\hat{\tilde{\omega}}_0^d(\theta^{-1}(\xi + 2\pi\lambda))}\ \overline{\tilde{m}^{d,q,h}\big(\theta^{-1}(\xi + 2\pi\lambda)\big)} \\
&= \sum_{\sigma=0}^{\theta-1} \sum_{h=1}^{D} m^{d,p,h}\big(\theta^{-1}(\xi + 2\pi\sigma)\big)\ \overline{\tilde{m}^{d,q,h}\big(\theta^{-1}(\xi + 2\pi\sigma)\big)} \\
&\quad \times \hat{\omega}_0^d\big(\theta^{-1}(\xi + 2\pi\sigma) + 2\pi\lambda\big)\,\overline{\hat{\tilde{\omega}}_0^d(\theta^{-1}(\xi + 2\pi\sigma) + 2\pi\lambda)} \\
&= \sum_{\sigma=0}^{\theta-1} \sum_{h=1}^{D} m^{d,p,h}\big(\theta^{-1}(\xi + 2\pi\sigma)\big)\ \overline{\tilde{m}^{d,q,h}\big(\theta^{-1}(\xi + 2\pi\sigma)\big)}\ . \blacksquare
\end{aligned}$$

Theorem 3.5. Suppose $\{\omega_n^d(x): n \ge 0, 1 \le d \le D\}$ and $\{\tilde{\omega}_n^d(x): n \ge 0, 1 \le d \le D\}$ are nonuniform multiwavelet Packets of multipilicity D with respect to a pair of biorthogonal scaling functions $\omega_0^d(x)$ and $\tilde{\omega}_0^d(x)$, respectively. Then, for $n \ge 0$, we have

$$\langle \omega_n^d(.), \tilde{\omega}_n^d(.-\lambda)\rangle = \delta_{0,\lambda}\,, \qquad \lambda \in \Lambda, 1 \le d \le D \qquad (23)$$

Proof. The proof it will be by induction, for $n = 0,1, \dots, \theta - 1$, (15) is true from (11), (14). Assume that it holds when $n < l;\ l > 0$. For $n = l$. Order $n = q + \theta p$, where $p \ge 0, 0 \le q \le \theta - 1\ \&\ p < n$. We have
$\langle \omega_p^d(.),\ \tilde{\omega}_p^d(.-\lambda)\rangle = \delta_{0,\lambda}\ \Leftrightarrow\ \sum_{d=1}^{D} \sum_{\lambda\in\Lambda} \hat{\omega}_p^d(\xi - \lambda)\, \overline{\hat{\tilde{\omega}}_p^d(\xi - \lambda)} = 1\ , \xi \in \mathbb{R}.$
And

$$\begin{aligned}
\langle \omega_n^d(.),\ \tilde{\omega}_n^d(.-\lambda)\rangle &= \langle \hat{\omega}_n^d(.),\ \hat{\tilde{\omega}}_n^d(.-\lambda)\rangle \\
&= \int_{\mathbb{R}} \hat{\omega}_{q+\theta p}^d(\xi)\ \overline{\hat{\tilde{\omega}}_{q+\theta p}^d(\xi)}\ e^{2\pi i\lambda\xi}\ d\xi \\
&= \int_{\mathbb{R}} \sum_{h=1}^{D} m^{d,q,h}(\xi/\theta)\, \hat{\omega}_p^h(\xi/\theta)\ \sum_{h=1}^{D} \overline{\tilde{m}^{d,q,h}(\xi/\theta)}\, \overline{\hat{\tilde{\omega}}_p^h(\xi/\theta)}\ e^{2\pi i\lambda\xi}\ d\xi \\
&= \sum_{\lambda\in\Lambda} \sum_{h=1}^{D} \int_{\theta([0,2\pi]+\lambda)} m^{d,q,h}\left(\frac{\xi}{\theta} + 2\pi\lambda\right) \hat{\omega}_p^h\left(\frac{\xi}{\theta} + 2\pi\lambda\right) \overline{\tilde{m}^{d,q,h}(\xi/\theta)}\ \overline{\hat{\tilde{\omega}}_p^h(\xi/\theta)}\ \ e^{2\pi i\lambda\xi}\ d\xi \\
&= \int_{\theta[0,2\pi]} \left(\sum_{\lambda\in\Lambda} \sum_{h=1}^{D} \hat{\omega}_p^h\left(\frac{\xi}{\theta} + 2\pi\lambda\right) \overline{\hat{\tilde{\omega}}_p^h(\frac{\xi}{\theta} + 2\pi\lambda)}\right) \times \left(\sum_{h=1}^{D} m^{d,q,h}\left(\frac{\xi}{\theta}\right) \overline{\tilde{m}^{d,q,h}(\xi/\theta)}\right)\ \ e^{2\pi i\lambda\xi}\ d\xi \\
&= \int_{\theta[0,2\pi]} \sum_{h=1}^{D} m^{d,q,h}\left(\frac{\xi}{\theta}\right) \overline{\tilde{m}^{d,q,h}(\xi/\theta)}\, e^{2\pi i\lambda\xi}\ d\xi \\
&= \int_{[0,2\pi]} \sum_{\sigma=0}^{\theta-1} \sum_{h=1}^{D} m^{d,q,h}\left(\frac{\xi}{\theta} + 2\pi\sigma\right) \overline{\tilde{m}^{d,q,h}(\frac{\xi}{\theta} + 2\pi\sigma)}\, e^{2\pi i\lambda\xi}\ d\xi
\end{aligned}$$

$$= \int_{[0,2\pi]} e^{2\pi i\lambda\xi}\ d\xi\ =\ \delta_{0,\lambda}. \qquad \blacksquare$$

Theorem 3.6. Suppose $\{\omega_n^d(x): n \geq 0, 1 \leq d \leq D\}$ and $\{\widetilde{\omega}_n^d(x): n \geq 0, 1 \leq d \leq D\}$ are Nonuniform multiwavelet Packets of multipilicity D with respect to a pair of biorthogonal scaling functions $\omega_0^d(x)$ and $\widetilde{\omega}_0^d(x)$, respectively. Then, for $n \geq 0$, we have

$$\langle \omega_{q_{1+\theta p}}^d(.), \widetilde{\omega}_{q_{2+\theta p}}^d(.-\lambda)\rangle = \delta_{0,\lambda}\ \delta_{q_1,q_2}\ , \lambda \in \Lambda, 0 \leq q_1, q_2 \leq \theta - 1, 1 \leq d \leq D \qquad (24)$$

Proof. Similar steps in above proof.

Theorem 3.7. If $\{\omega_n^d(x): n \geq 0, d = 1, \ldots, D\}$ and $\{\widetilde{\omega}_n^d(x): n \geq 0,\ d = 1, \ldots, D\}$ are basic nonuniform multiwavelet packets with respect to a pair of biorthogonal multiscaling functions $\omega_0^d(x)$ and $\widetilde{\omega}_0^d(x)$, respectively. Then, for $m, n \geq 0$, we have

$$\langle \omega_m^d(.), \widetilde{\omega}_n^d(.-\lambda)\rangle = \delta_{m,n}\ \delta_{0,\lambda} \qquad (25)$$

Proof. For $m = n$, the result (25) follows by Theorem 3.5. When $m \neq n$, and $0 \leq m, n \leq \theta - 1$ $0 \leq \ell$, the result (25) can be get from Theorem 3.6. Assuming that $m \neq n$, and at least one of $\{m, n\}$ does not lies in $0,1,\ldots,\theta - 1$, then we can rewrite m, n as $m = q_1 + \theta p_1, n = q_2 + \theta p_2$, where $p_1, p_2 \geq 0$, $0 \leq q_1, q_2 \leq \theta - 1$.

Case 1. If $p_1 = p_2$, then $q_1 \neq q_2$. Therefore, Eq. (25) follows by (11),(15), Lemma and (22), i.e.,

$$\langle \omega_m^d(.), \omega_n^d(.-\lambda)\rangle = \langle \omega_{q_1+\theta p_1}^d(.), \omega_{q_2+\theta p_2}^d(.-\lambda)\rangle$$

$$= \langle \widehat{\omega}_{q_1+\theta p_1}^d(.), \widehat{\widetilde{\omega}}_{q_2+\theta p_2}^d(.-\lambda)\rangle$$

$$= \int_{\mathbb{R}} \widehat{\omega}_{q_1+\theta p_1}^d(.)\ \overline{\widehat{\widetilde{\omega}}_{q_2+\theta p_2}^d(.-\lambda)}\ e^{2\pi i\lambda\xi}\ d\xi$$

$$= \int_{\mathbb{R}} m^{d,q_1,h}(\theta^{-1}\xi)\ \widehat{\omega}_{p_1}^d(\theta^{-1}\xi)\ \overline{\widehat{\widetilde{\omega}}_{p_2}^d(\theta^{-1}\xi)}\ \overline{\widetilde{m}^{d,q_2,h}(\theta^{-1}\xi)}\, e^{2\pi i\lambda\xi}\, d\xi$$

$$= \sum_{\lambda\in\Lambda} \int_{\theta([0,2\pi]+\lambda)} m^{d,q_1,h}(\theta^{-1}\xi)\ \widehat{\omega}_{p_1}^d(\theta^{-1}\xi)\ \overline{\widehat{\widetilde{\omega}}_{p_2}^d(\theta^{-1}\xi)}\ \overline{\widetilde{m}^{d,q_2,h}(\theta^{-1}\xi)}\, e^{2\pi i\lambda\xi}\, d\xi$$

$$= \int_{\theta[0,2\pi]} \left(\sum_{\lambda\in\Lambda} \widehat{\omega}_{p_1}^d(\theta^{-1}(\xi+\lambda))\ \overline{\widehat{\widetilde{\omega}}_{p_2}^d(\theta^{-1}(\xi+\lambda))}\right) m^{d,q_1,h}(\theta^{-1}\xi)\ \overline{\widetilde{m}^{d,q_2,h}(\theta^{-1}\xi)}\, e^{2\pi i\lambda\xi} d\xi$$

$$= \int_{[0,2\pi]} \sum_{\sigma=0}^{\theta-1} m^{d,q_1,h}(\theta^{-1}(\xi+2\pi\sigma))\ \overline{\widetilde{m}^{d,q_2,h}(\theta^{-1}(\xi+2\pi\sigma))}\, e^{2\pi i\lambda\xi}\ d\xi$$

$$= \int_{[0,2\pi]} \delta_{q_1,q_2}\ e^{2\pi i\lambda\xi}\ d\xi\ =\ \delta_{0,\lambda} = 0$$

Case 2. If $p_1 \neq p_2$, order $p_1 = q_3 + \theta\, p_3$, $p_2 = q_4 + \theta\, p_4$, where $p_3, p_4 \geq 0$ and $0 \leq q_3, q_4 \leq \theta - 1$. If $p_3 = p_4$, then $q_3 \neq q_4$. Similar to Case 1, (25) can be established. When $p_3 \neq p_4$, we order $p_3 = q_5 + \theta\, p_5, p_4 = q_6 + \theta\, p_6$, where $p_5, p_6 \geq 0$ and $0 \leq q_5, q_6 \leq \theta - 1$. Thus, after taking finite steps (denoted by k), we obtain $0 \leq p_k, p_{2k} \leq \theta - 1$ and $0 \leq p_k, q_{2k} \leq \theta - 1$. If $p_k = p_{2k}$, then $p_k \neq q_{2K}$.Similar to the Case 1, (25) follows. If $p_k \neq q_k$

, then

$$\langle \omega_{p_k}^d(.), \widetilde{\omega}_{p_{2k}}^d(.)\rangle = 0,\quad \lambda \in \Lambda \iff \sum_{\lambda\in\Lambda} \widehat{\omega}_{p_k}^d(\xi-\lambda)\ \overline{\widehat{\widetilde{\omega}}_{p_{2k}}^d(\xi-\lambda)} = 0$$

Furthermore, we obtain

$$\langle \omega_m^d(.), \widetilde{\omega}_n^d(.-\lambda)\rangle = \langle \widehat{\omega}_m^d(.), \widehat{\widetilde{\omega}}_n^d(.-\lambda)\rangle$$

$$= \langle \widehat{\omega}_{q_1+\theta p_1}^d(.), \widehat{\widetilde{\omega}}_{q_2+\theta p_2}^d(.-\lambda)\rangle$$

$$= \int_{\mathbb{R}} \widehat{\omega}^{d}_{q_1+\theta p_1}(.)\, \overline{\widehat{\widetilde{\omega}}^{d}_{q_2+\theta p_2}(.-\lambda)}\; e^{2\pi i\lambda \xi}\, d\xi$$

$=\int_{\mathbb{R}} m^{d,q_1,h}\left(\theta^{-1}\xi\right) m^{d,q_3,h}\left(\theta^{-2}(\xi +\right.$
2□□)□ □3□□−2□□□4□□−2□□□,□2,ℎ□−1□□□,□4,ℎ□−1□□2□□□□□□

$$= \int_{\mathbb{R}} \left(\prod_{r=1}^{k} m^{d,q_r,h}(\theta^{-r}\xi)\right) \widehat{\omega}^{d}_{p_k}(\theta^{-k}\xi)\, \overline{\widehat{\widetilde{\omega}}^{d}_{p_{2k}}(\theta^{-k}\xi)}\, \left(\prod_{r=1}^{2k} m^{d,q_{r+1},h}(\theta^{-r}\xi)\right) e^{2\pi i\lambda\xi}\, d\xi$$

$$= \sum_{\lambda\in\Lambda} \int_{\theta^{k}([0,2\pi]+\lambda)} \left(\prod_{r=1}^{k} m^{d,q_r,h}(\theta^{-r}\xi)\right) \left(\widehat{\omega}^{d}_{p_k}(\theta^{-k}\xi)\, \overline{\widehat{\widetilde{\omega}}^{d}_{p_{2k}}(\theta^{-k}\xi)}\right) \left(\prod_{r=1}^{2k} m^{d,q_{r+1},h}(\theta^{-r}\xi)\right) e^{2\pi i\lambda\xi}\, d\xi$$

$$= \int_{\theta^{k}[0,2\pi]} \left(\prod_{r=1}^{k} m^{d,q_r,h}(\theta^{-r}\xi)\right) \left(\prod_{r=1}^{2k} m^{d,q_{r+1},h}(\theta^{-r}\xi)\right) \sum_{\lambda\in\Lambda} \left(\widehat{\omega}^{d}_{p_k}(\theta^{-k}(\xi + \lambda))\, \overline{\widehat{\widetilde{\omega}}^{d}_{p_{2k}}(\theta^{-k}(\xi+\lambda))}\right) e^{2\pi i\lambda\xi}\, d\xi$$

$= 0$ ■

REFERENCES

[1]. Atlouba, N. A. S., Mittal, S. and Paul, A., (2015), Nonuniform Multiwavelet Packets associated ith Nonuniform Multiresolution Analysis with Multiplicity D, J. Progressive Research in Mathematics, 3(3), 192-202.

[2]. Behera, B., (2007). Wavelet packets associated with nonuniform multiresolution analysis, J. Math. Anal. Appl, 328, 1237-1246.

[3]. Behera, B., (2001) Multiwavelet packets and frame packets of L2(Rd), Proc. Ind. Acad. Sci., 111, 439-463.

[4]. Calogero, A., & G. Garrig´os, (2001) A characterization of wavelet families arising from biorthogonal MRA's of multiplicity d, J. Geom. Anal. 11(2), 187–217.

[5]. Coifman R., & Meyer, Y., (1989). Orthonormal wave packet bases, (Yale University)

[6]. Coifman R., Meyer, Y. and Wickerhauser, M. V., (1992). Wavelet analysis and signal procesing, in: Wavelets and Their Applications (eds) M B Ruskai et al (Boston: Jones and Bartlett), 153-178.

[7]. Coifman, R., Y. Meyer and M. V. Wickerhauser, Size properties of wavelet packets, in: Wavelets and Their Applications (eds) M B Ruskai et al (Boston: Jones and Bartlett) (1992) 453-470.

[8]. F. A. Shah, (1998). Biorthogonal Wavelet Packets Associated with Nonuniform Multiresolution, J. Information and Computing Science,9(1), (2014) 011-021.

[9]. Gabardo, J. P. and Nashed, M. Z., Nonuniform multiresolution analyses and spectral pairs, J. Funct. Anal., 158(1), 209–241.

[10]. Gabardo, J.-P. & Nashed, M. Z., (1998). An analogue of Cohen's condition for nonuniform multiresolution analyses, Wavelets, multiwavelets, and their applications (San Diego, CA, 1997), 41–61, Contemp. Math., 216, Amer. Math. Soc., Providence, RI.

[11]. Gabardo, J. P. & Yu, X., (2006). Wavelets associated with nonuniform multiresolution analyses and one-dimensional spectral pairs, J. Math. Anal. Appl., 323(2) 798–817. 13.

[12]. Long, R. & Chen, W. (1997). Wavelet basis packets and wavelet frame packets, J. Fourier Anal. Appl. 3(3) 239256

[13]. Mittal, S. and Shukla, N. K., Generalized nonuniform multiresolution analyses, preprint.

[14]. Mittal, S., Shukla, N. K. and N. Atlouba, A. S., Nonuniform multiresolution analyses with multiplicity D, preprint.

[15]. R. R. Coifman, Y. Meyer, S. Quake, and M. V. Wickerhauser, (1990). Signal processing and compression with wavelet packets, Technical Report, Yale University.

[16]. S. G. Mallat, (1989). Multiresolutions and wavelet orthonormal bases of $L^2(\mathbb{R})$, ChaosAmer. Math. Soc. 315, 130-137.

[17]. Wang, X. (1995). The study of wavelets from the properties of their Fourier transforms, Thesis (Ph.D.)-Washington University in St. Louis, 138 pp.

[18]. Yu, X. (2005). Wavelet sets, integral self-affine tiles and nonuniform multiresolution analyses, Thesis(Ph.D.)–McMaster University (Canada), 145 pp.