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International
Journal
Of Modern Engineering Research (IJMER)

ON SUMS AND PRODUCTS OF q-k-EP MATRICES

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ABSTRACT:- Necessary and Sufficient conditions for the sums and products of q-k-EP matrices of rank r to be q-k-EP in minkowski space *m* are discussed. Also equivalent conditions for the product of q-k-EPblock matrices to be q-k-EP are established.As an application it is shown that the sum and parallel sum of parallel summable q-k-EP matrices are q-k-EP and we have shown that a block matrix in Minkowski space can be expressed as a product of q-k-EP matrices in *m*.

Keywords:- q-k-EP matrices, Range symmetric matrices, Minkowski space,etc.,
AMS Mathematical classification: 15A57

I. INTRODUCTION

Throughout we shall deal with H_{nxn} , the space of nxn complex matrices. Let H_n be the space of complex n - tuples. For $A \in H_{nxn}$, let A^T and A^* denote the transpose, conjugate transpose of A. Let A^- be a generalized inverse ($AA^-A = A$) and A be the Moore-Penrose inverse of A [5]. A matrix A is called EP_r if ρ(A)= r and N(A)=N(A*) or R (A)=R(A*) where ρ(A) denotes the rank of A; N(A) and R(A) denote the null space and range space of A respectively. Through let 'k' be a fixed product of disjoint transpositions in S_n={1,2,...,n} and K be the associated permutation matrix. A matrix A =(a_{ij}) ∈ C_{nxn} is said to be k-hermitian if $a_{ij}=\bar{a}_{k(j),k(i)}$ for i,j=1,2,...,n. A theory for k-hermitian matrices is developed in [1].

For $x = (x_1, x_2, , \dots, x_n)^T \in H_n$. Let us define the function, $K(x) = (x_{k(1)}, x_{k(2)}, \dots, x_{k(n)})T \in H_n$. A matrix $A \in H_{nxn}$ said to be q-k-EP if it satisfies the condition Ax =0 ⇔A*k(x)=0 or equivalently N(A)=N(A *K). In addition to that, A is q-k-EP KA is EP or AK is EP and A is q-k-EP A * is q-k-EP. Moreover, A is said to be $q\text{-}k-EP_r$, if A is q-k-EP and of rank r. For further properties of q-k-EP matrix one may refer[4]. Let G be the Minkowski metric tensor defined by $Gx = (x_1, -x_2, -x_3, \dots, -x_n)^T$ for $(x_1, x_2, , \dots, x_n) \in H_n$. Clearly the Minkowski metric matrix.

Minkowski inner product on C_n is defined by (u,v)=[u,Gv], where [.,.] denotes the conventional Hilbert space inner product. A space with Minkowski inner product is called a Minkowski space and denoted as m. With respect to the Minkowski inner product, since (Ax,y)=(x, A˜y),A =GA*G is called the Minkowski adjoint of the matrix $A \in H_{nxn}$ and A* is the usual Hermitian adjoint. In this paper we give necessary and sufficient conditions for sums ofq-k-EP matrix to be q-k-EP. As an application it is shown that sum and parallel summable q-k-EP matrices are q-k-EP.

II. SUMS OF K-EP MATRICES

Lemma: 2.1

Let $A_1, A_1, \dots, A_1, \in H_{nxn}$ and let $A = \sum_{i=1}^{m} A_i$. Consider the following conditions:

(a) $N(A) \subseteq N(A_i)$ for $i = 1,2, \dots, m$;

(b) $N(A) = \bigcap_{i=1}^{m} N(A_i)$;

(c) $\rho(A) = \rho\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$;

(d) $\sum_{i=1}^{m} \sum_{j=1}^{m} {A_i}^* A_j = 0$;

(e) $\rho(A) = \sum_{i=1}^{m} \rho(A_i)$.

Then the following statements hold:

(i) Conditions (a) , (b) and (c) are equivalent.

(ii) Condition (d) implies (a), but (a) does not implies (d).

(iii) Condition (e) implies (a), but (a) does not implies (e).

Proof:[11]

(i) (a)⇔(b)⇔(c):

$N(A) \subseteq N(A_i)$ for each I $\Rightarrow N(A) \subseteq \cap N(A_i)$.

Since $N(A) \subseteq N(\sum A_i) \supseteq N(A_1) \cap N(A_2) \ldots \cap N(A_m)$, it follows that $N(A) \supseteq \cap N(A_1)$.

Always $\bigcap_{i=1}^{m} N(A_i) \subseteq N(A)$. Hence $N(A) = \bigcap_{i=1}^{m} N(A_i)$;

Thus (b) holds.

Now, $N(A) = \bigcap_{i=1}^{m} N(A_i) = N\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$

Therefore, $\rho(A) = \rho\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$ and (c) holds.

Conversely, $\rho(A) = \rho\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$ and $N\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} = \bigcap_{i=1}^{m} N(A_i) \subseteq N(A) \Rightarrow N(A) = \bigcap_{i=1}^{m} N(A_i)$

And (b) holds.

Hence $N(A) \subseteq N(A_i)$ for each i and (a) holds.

(ii) (d)⇒ (a):

Since $\sum_{i \neq j} A_i{}^* A_j = 0$

$A^*A = (\sum A_i)^*(\sum A_i)$

$= (\sum A_i{}^*)(\sum A_i)$

$= \sum A_i{}^* A_i$

$N(A) = N(A^*A) = N(\sum A_i{}^* A_i)$

$$= N\left(\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}^* \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}\right)$$

$$= N\left(\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}\right)$$

$= N(A_1) \cap N(A_2) \ldots \cap N(A_m)$

$= \bigcap_{i=1}^{m} N(A_i)$.

Hence $N(A) \subseteq N(A_i)$ for each i and (a) holds.

(a)⇏(d):

Let us consider the following example,

Let $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$A_1 + A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Clearly, $N(A_1 + A_2) \subseteq N(A_1)$. Also $N(A_1 + A_2) \subseteq N(A_2)$. But $A_1{}^* A_2 + A_2{}^* A_1 \neq 0$.

(iii)(e) ⇒(a):

If rank is additive, that is $\rho(A) = \sum \rho(A_i)$, then by [3],

$R(A_i) \cap R(A_j) = \{0\}, i \neq j \Rightarrow N(A) \subseteq N(A_i)$ for each i and (a) holds.

(a)⇏ (e): Consider the example,

Let $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

$A_1 + A_2 = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$.

Here, $N(A_1 + A_2) \subseteq N(A_1)$ and $N(A_1 + A_2) \subseteq N(A_2)$.

But $\rho(A_1 + A_2) \neq \rho(A_1) + \rho(A_2)$.

Theorem 2.2

Let $A_1, A_2, \ldots, A_m \in H_{nxn}$ be q-k-EP matrices. If any one of the conditions (a) to (e) of Lemma 2.1 holds, then $A = \sum_{i=1}^{m} A_i$ is q-k-EP.

Proof:

Since each A_i is q-q-k-EP, $N(A_i) = N(A_i{}^* K)$ for each i.

Now, $N(A) \subseteq N(A_i)$ for each i

$\Rightarrow N(A) \subseteq \bigcap_{i=1}^{m} N(A_i{}^* K) \subseteq N(A^* K)$

And $\rho(A) = \rho(A^* K)$. Hence $N(A) = (N(A^* K)$. Thus A is q-k-EP. Hence the theorem.

Remark 2.3

In particular, if A is non-singular the conditions automatically hold and A is q-k-EP. Theorem 2.2 fails if we relax the conditions on the A_i's.

Example 2.4

Consider $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $A_2=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then the associated permutation matrix $K=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Thus, $KA_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is EP.

Therefore, A_1 is q-k-EP.

$KA_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is not EP. Therefore A_2 is not q-k-EP.

$A_1 + A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $K(A_1 + A_2) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ which is not EP.

Therefore, $(A_1 + A_2)$ is not q-k-EP. However,

$N(A_1 + A_2) \subseteq N(A_1{}^*K) \subseteq N(A_1)$ and $N(A_1 + A_2) \subseteq N(A_2{}^*K) \subseteq N(A_2)$.

Moreover, $\rho\left(\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\right) = \rho(A_1 + A_2)$.

Remark 2.5

Theorem 2.2 fails if we relax the condition that A_i's are q-q-k-EP.

For , let $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and let the associated permutation matrix be $K=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

$KA_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is not EP.

Therefore, A_1 is not q-k-EP.

$KA_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is not EP.

Therefore, A_2 is not q-k-EP.

$A_1 + A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$ and $K(A_1 + A_2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ which is not EP.

Therefore, $(A_1 + A_2)$ is not q-k-EP. But $A_1{}^*A_2 + A_2{}^*A_1 = 0$.

Remark 2.6

The conditions given in Theorem 2.2 are only sufficient for the sum of q-k-EP matrices to be q-k-EP, but not necessary is illustrated in the following example.

Example 2.7

Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_2=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and the associated permutation matrix $K=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A_1 and A_2 are $k\text{-}EP_2$. The conditions in Theorem 2.2 does not hold. However $(A_1 + A_2)$ is q-q-k-EP.

Remark 2.8

If A_1 and A_2 are q-k-EP matrices, then by Theorem 2.4(p.221,[4]),

$A_1{}^* = H_1KA_1K$ and $A_2{}^* = H_2KA_2K$ where H_1 and H_2 are non-singular nxn matrices. If $H_1 = H_2$, then

$A_1{}^* + A_2{}^* = H_1K(A_1 + A_2)K \Rightarrow (A_1 + A_2)^* = H_1K(A_1 + A_2)K \Rightarrow A_1 + A_2$ is q-k-EP.

If $(H_1 - H_2)$ is non-singular, then above conditions are also necessary for the sum of q-k-EP matrices to be q-k-EP is given in the following theorem.

Theorem 2.9 [11]

Let K be the permutation matrix associated with the fixedntransposition 'k'. Let $A_1{}^* = H_1KA_1K$ and $A_2{}^* = H_2KA_2K$ such that $(H_1 - H_2)$ is non-singular. Then $A_1 + A_2$ is q-k-EP if and only if $N(A_1 + A_2) \subseteq N(A_i)$ for some (and hence both) $i \in \{1,2\}$.

Proof: Since $A_1{}^* = H_1KA_1K$ and $A_2{}^* = H_2KA_2K$, by Remark 2.8, A_1 and A_2 are q-k-EP matrices. Since, $N(A_1 + A_2) \subseteq N(A_2)$ by theorem 2.2, $(A_1 + A_2)$ is q-k-EP. Conversely, let us assume that $A_1 + A_2$ is q-k-EP. By Remark 2.8, there exists a non-singular matrix G such that

$(A_1 + A_2)^* = GK(A_1 + A_2)K$

$\Rightarrow A_1{}^* + A_2{}^* = GK(A_1 + A_2)K$

$\Rightarrow H_1KA_1K + H_2KA_2K = GK(A_1 + A_2)K$

$\Rightarrow (H_1KA_1 + H_2KA_2)K = GK(A_1 + A_2)K$

$\Rightarrow (H_1K - GK)A_1 = (GK - H_2K)\, A_2$

$\Rightarrow (H_1 - G)KA_1 = (G - H_2)K A_2$
$\Rightarrow LKA_1 = M K A_2$, where $L = (H_1 - G)$, $M = (G - H_2)$
Now $(L+M)(KA_1) = LKA_1 + MKA_1$
$= MKA_2 + MKA_1$
$= MK(A_1 + A_2)$
And $(L+M)(KA_2) = LK(A_1 + A_2)$
By hypothesis, $L+M = H_1 - G + G - H_2 = H_1 - H_2$ is non-singular.
Therefore, $N(A_1 + A_2) \subseteq N(A_1)$ and $N(A_1 + A_2) \subseteq N(A_2)$. Hence the theorem.

Remark 2.10

The condition $H_1 - H_2$ to be non-singular is essential in Theorem 2.9 is illustrated in the following example.

Example 2.11

Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ are both k-EP matrices for the associated permutation matrix, $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Further $A_1^* = A_1 = KA_1K$ and $A_2^* = A_2 = KA_2K \Rightarrow H_1 = H_2 = I$.
$(A_1 + A_2) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is also q-k-EP.
But $N(A_1 + A_2) \not\subset N(A_1)$ or $N(A_1 + A_2) \not\subset N(A_2)$. Thus Theorem 2.9 fails.

III. PARALLEL SUMMABLE Q-K-EP MATRICES

In this section we shall show that sum and parallel sum of parallel summable (p.s) q-k-EP matrices are q-k-EP. First we shall give the definition and some properties of parallel summable matrices as in (p.188, [5]).

Definition: 3.1

A_1 and A_2 are said to be parallel summable (p.s) if $N(A_1 + A_2) \subseteq N(A_2)$ or $N(A_1 + A_2)^* \subseteq N(A_2^*)$ or equivalently $N(A_1 + A_2) \subseteq N(A_1)$ and $N(A_1 + A_2)^* \subseteq N(A_1^*)$.

Definition: 3.2

If A_1 and A_2 are parallel summable(p.s) then parallel sum A_1 and A_2 denoted by $A_1 \mp A_2$ is defined as $A_1 \mp A_2 = A_1(A_1 + A_2)^- A_1$. The product $A_1(A_1 + A_2)^- A_1$ is invariant for all choices of generalized inverse $(A_1 + A_2)^-$ of $(A_1 + A_2)$ under the conditions that A_1 and A_2 are parallel summable(p.188,[5]).

Properties: 3.3

Let A_1 and A_2 be a pair of parallel summable(p.s) matrices. Then the following hold:

P.1 $A_1 \mp A_2 = A_2 \mp A_1$
P.2 A_1^* and A_2^* are p.s and $(A_1 \mp A_2)^* = A_1^* \mp A_2^*$
P.3 If U is non-singular then UA_1 and UA_2 are p.s and
$(UA_1 \mp UA_2) = U(A_1 \mp A_2)$
P.4 $R(A_1 \mp A_2) = R(A_1) \cap R(A_2)$
$N(A_1 \mp A_2) = R(A_1) + R(A_2)$
P.5 $(A_1 \mp A_2) \mp A_3 = A_1 \mp (A_2 \mp A_3)$

Lemma: 3.4[11]

Let A_1 and A_2 be q-k-EP matrices. Then A_1 and A_2 are p.s if and only if
$N(A_1 + A_2) \subseteq N(A_1)$ for some (and hence both) $i \in \{1,2\}$.

Proof:

Let A_1 and A_2 be a pair of parallel summable(p.s) matrices
$\Rightarrow N(A_1 + A_2) \subseteq N(A_1)$ follows from the Definition 3.1, conversely if $N(A_1 + A_2) \subseteq N(A_1)$, then
$N(KA_1 + KA_2) \subseteq N(KA_1)$. Also $N(KA_1 + KA_2) \subseteq N(KA_2)$.
Since A_1 and A_2 are q-k-EP matrices, KA_1 and KA_2 are EP matrices,
$N(KA_1 + KA_2) \subseteq N(KA_1)$, and $N(KA_1 + KA_2) \subseteq N(KA_2)$. Therefore
$KA_1 + KA_2$ is EP.
Hence $N(KA_1 + KA_2)^* = N(KA_1 + KA_2)$
$= N(KA_1) \cap N(KA_2)$
$= N(KA_1)^* \cap N(KA_2)^*$
Therefore, $N(KA_1 + KA_2)^* \subseteq N(KA_1)^*$, $N(KA_1 + KA_2)^* \subseteq N(KA_2)^*$
Also $N(KA_1 + KA_2) \subseteq N(KA_1)$
$\Rightarrow N(K(A_1 + A_2) \subseteq N(KA_1)$
$\Rightarrow N(A_1 + A_2) \subseteq N(A_1)$
Similarly, $N(A_1 + A_2)^* \subseteq N(A_1)^*$.
Therefore, A_1 and A_2 be a pair of parallel summable(p.s) matrices. Hence the theorem.

Remark: 3.5

Lemma 3.4 fails if we relax the condition that $A_1$and $A_2$are q-k-EP matrices. Let $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and A_2=$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let the associated permuatation matrix K =$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

A_1 is q-k-EP. A_2 is not q-k-EP.
$N((A_1 + A_2) \subseteq N(A_1)$ and $N((A_1 + A_2) \subseteq N(A_2)$,
But $N(A_1 + A_2)^* \not\subset N(A_1)^*$, $N(A_1 + A_2)^* \not\subset N(A_2)^*$.
Hence $A_1$and $A_2$are not parallel summable(p.s) matrices.

Theorem: 3.6

Let$A_1$and A_2 be a pair of parallel summable(p.s) q-k-EP matrices. Then $A_1 \mp A_2$ and $(A_1 + A_2)$ are q-k-EP.

Proof:

Since $A_1$and A_2 be a pair of parallel summable(p.s) q-k-EP matrices, by Lemma 3.4,
$N(A_1 + A_2) \subseteq N(A_1)$ and $N(A_1 + A_2) \subseteq N(A_2)$. Hence,
$N(K(A_1 + A_2) \subseteq N(KA_1)$ and $N(K(A_1 + A_2) \subseteq N(KA_2)$
$N(KA_1 + KA_2) \subseteq N(KA_1)$ and $N(KA_1 + KA_2) \subseteq N(KA_2)$
Therefore, $KA_1 + KA_2$=$K(A_1 + A_2)$ is EP. Then $A_1 + A_2$ is q-k-EP. Since $A_1$and A_2 be a pair of parallel summable(p.s) q-k-EP matrices, KA_1 and KA_2 are p.s EP matrices.
Therefore,
$R(KA_1)^* = R(KA_1)$ and $R(KA_2)^* = R(KA_2)$
$R(KA_1 \mp KA_2)^* = R((KA_1)^* \mp (KA_2)^*)$ [By P.2]
$= R(KA_1)^* \cap R(KA_2)^*$ [By P.4]
$= R(KA_1) \cap R(KA_2)$[since $KA_1$and $KA_2$are EP].
$= R(KA_1 \mp KA_2)$.
Thus, $KA_1 \mp KA_2$is EP $\Rightarrow K(A_1 \mp A_2)$ is EP$\Rightarrow (A_1 \mp A_2)$is q-k-EP. Thus $K(A_1 \mp A_2)$ is q-k-EP whenever $A_1$and A_2 are q-k-EP. Hence the theorem.

Corollary:

Let $A_1$and A_2 are q-k-EP matrices such that $N(A_1 + A_2) \subseteq N(A_2)$. If A_3 is q-k-EP commuting with both A_1andA_2, then $A_3(A_1 + A_2)$ and $A_3(A_1 \mp A_2)$=$(A_3A_1 \mp A_3A_2)$ are q-k-EP.

Proof:

$A_1$and A_2 are q-k-EP with $N(A_1 + A_2) \subseteq N(A_2)$.
By Theorem 2.2, $(A_1 + A_2)$is q-k-EP. NowKA_1, KA_2 and $K(A_1 + A_2)$ are EP. Since A_3 commutes with $A_1, A_2$and $(A_1 + A_2), KA_3$ commutes with $KA_1, KA_2$and $K(A_1 + A_2)$ and by Theorem (1.3) of [2], $K(A_3A_1), K(A_3A_2)$ and $KA_3(A_1 + A_2)$ are EP. Therefore, $(A_3A_1), (A_3A_2)$ and $A_3(A_1 + A_2)$ are q-k-EP. Noe by Theorem 3.6, $(A_3A_1 \mp A_3A_2)$ are q-k-EP By P.3(Properties 3.3).
$KA_3(A_1 \mp A_2)$=$K(A_3A_1 \mp A_3A_2)$.
Since $(A_3A_1 \mp A_3A_2)$ is q-k-EP. $K(A_3A_1 \mp A_3A_2)$ is EP $\Rightarrow KA_3(A_1 \mp A_2)$ is EP. $A_3(A_1 \mp A_2)$ is q-k-EP.
Hence the corollary.

IV. PRODUCT OF Q-K-EP MATRICES IN MINKOWSKI SPACE

Lemma: 4.1

Let A and B be matrices in *m*. Then $N(A^*) \subseteq N(B^*) \Leftrightarrow N(A^-) \subseteq N(B^-)$.

Theorem 4.2[8]

For $A, B, C \in H_{mxn}$, then the following are equivalent:

(1) CA^-B is invariant for every $A^- \in H_{nxm}$.

(2) $N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$

(3) $C = CA^-A$ and $B = AA^-B$ for every $A^- \in \{1\}$

Definition: 4.3 [8]

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an nxn matrix. A generalized schur complement of A in M denoted by M|A is defined as $D - CA^-B$, where A^- is a generalized inverse of A.

Definition: 4.4[10]

A matrix $A \in H_{nxn}$, is said to be q-k-EP in *m* if and only if $N(A) = N(A^-K)$.

Lemma: 4.5[12]

For $A \in H_{nxn}$, the following are equivalent

(1) A is q-k-EP

(2) KA is EP

(3) AK is EP

(4) $KA^{\dagger}A = AA^{\dagger}K$

Lemma: 4.6[7]

For $A \in H_{nxn}$, the following are equivalent

(1) A is q-k-EP in m.

(2) GA is q-k-EP

(3) AG is q-k-EP

Theorem: 4.7[8]

Let Mbe of the form (2.1), with ρ(M)=ρ(A)=r then M is q-k-EP matrix in *m* with $k = k_1k_2 \Leftrightarrow$A is q-$k_1$-EP in m and $CA^{\dagger}K_1 = -G_1(A^{\dagger}BK_2)^{\sim}$

Theorem: 4.8[12]

Let A and B be q-k-EP matrices in *m* of rank r and AB of rank r. Then AB is q-k-EP matrix in *m* of rank r if and only if $N(A) = N(B)$.

Proof:

AB is q-k-EP matrix in m of rank r

$\Rightarrow N(AB) = N(AB)^{\sim}K$ (by Definition 4.4)

$\Rightarrow N(B) = N(B^{\sim}A^{\sim})K$, since ρ(B)=ρ(AB)=r (by P.1)

$\Rightarrow N(B) \subseteq N(A^{\sim})K$

$\Rightarrow N(B) \subseteq N(A)$ (by Definition 4.4)

$\Rightarrow N(B) = N(A)$, (since ρ(A)=ρ(B)=r)

Conversely, let $N(A) = N(B)$. To prove that AB is q-k-EP in *m*. Clearly $N(AB) \subseteq N(B)$. Since ρ(B)=ρ(AB)=r, we get $N(AB) = N(B)$. (4.1)

$N((AB)^{\sim}K) = N(B^{\sim}A^{\sim})K \subseteq N(A^{\sim}K) = N(A)$ (by Definition 4.4)

Now, $N(A) = N(A^{\sim}K) \Rightarrow \rho(A^{\sim}K) = \rho(A) = r$

$N(B) = N(B^{\sim}K) \Rightarrow \rho(B^{\sim}K) = \rho(B) = r$

$N((AB)^{\sim}K) \subseteq N(A)$

$\rho((AB)^{\sim}K) = \rho((AB)^{\sim}) = \rho(AB) = r$

Hence $N((AB)^{\sim}K) = N(A)$, since $\rho(A) = r$ (4.2)

From (4.1) and (4.2) we get,

$N(AB) = N((AB)^{\sim}K)$, since $N(A) = N(B)$

Thus AB is q-k-EP in *m*.

Theorem: 4.9

Let A and B and AB be q-k-EP matrices in *m* of rank r and BA of rank r. Then BA is q-k-EP_r matrix in *m*.

Proof:

Let A and B and AB be q-k-EP matrices in *m* of rank r and BA of rank r.

We claim BA is q-k-EP_r in *m*.

$N(BA) \subseteq N(A)$.

$\rho(BA) = \rho(A) = r$

Therefore $N(B^{\sim}A^{\sim})K \subseteq N(A^{\sim}K) = N(A)$ (4.3)

$N(AB) \subseteq N(B)$.

$\rho(AB) = \rho(B) = r$

Therefore $N(AB) = N(B)$, (4.4)

By Theorem (4.8), $N(A) = N(B)$

Hence $N(AB) = N(BA)$. (4.5)

Also, $N((BA)^{\sim}K) = N(A^{\sim}B^{\sim})K \subseteq N(B^{\sim}K) = N(B)$

$N((BA)^{\sim}K) \subseteq N(B) = N(AB)$

$N((BA)^{\sim}K) \subseteq N(AB)$

$\rho((BA)^{\sim}K) = \rho((BA)^{\sim}) = \rho(BA) = \rho(A) = r$

$N((BA)^{\sim}K) = N(AB)$ (4.6)

From (3.5) and (3.6) it follows that $N(BA) = N((BA)^{\sim}K)$

Therefore, BA is q-k-EP_r matrix in *m*.

Lemma: 4.10[12]

For complex matrices A and B, $N(A^{*}K) \subseteq N(B^{*}K)$ if and only if $N(A^{\sim}K) \subseteq N(B^{\sim}K)$

Proof: Let us assume that $N(A^*K) \subseteq N(B^*K)$ we need to prove $N(A^{\sim}K) \subseteq N(B^{\sim}K)$
Let us choose $x \in N(A^{\sim}K) \Rightarrow A^{\sim}Kx = 0$
$\Rightarrow GA^*GKx = 0$
$\Rightarrow A^*GKx = 0$
$\Rightarrow A^*KKGKx = 0$ (by P.2)
$\Rightarrow A^*Ky = 0$,where $y = KGKx$and hence $Ky = GKx$
$\Rightarrow y \in N(A^*K) \subseteq N(B^*K) \Rightarrow B^*Ky = 0$
$\Rightarrow B^*GKx = 0$
$\Rightarrow GB^*GKx = 0$
Hence, $B^{\sim}Kx = 0$, $x \in N(B^{\sim}K)$
Thus $N(A^{\sim}K) \subseteq N(B^{\sim}K)$
Conversely, let us assume that $N(A^{\sim}K) \subseteq N(B^{\sim}K)$
We need to prove that $N(A^*K) \subseteq N(B^*K)$
Let us choose $x \in N(A^*K) \Rightarrow A^*Kx = 0$
$\Rightarrow GA^*GGKx = 0$
$\Rightarrow A^{\sim}GKx = 0$
$\Rightarrow A^{\sim}Ky = 0$, $\Rightarrow y = KGKx$
$\Rightarrow y \in N(A^{\sim}K) \subseteq N(B^{\sim}K)$
$\Rightarrow B^{\sim}Ky = 0$
$\Rightarrow GB^*GKy = 0$
$\Rightarrow GB^*GGKx = 0$
$\Rightarrow B^*Kx = 0$
$\Rightarrow x \in N(B^*K)$
Thus $N(A^*K) \subseteq N(B^*K)$. Hence the result.

REFERENCES

[1]. R.D.Hill and S.R. Waters, On k-Real and k-Hermitian matrices, Linear Algebra.Appl. 169(1992), 17-29.
[2]. I.J.Katz and M.H.Pearl, On EPr and normal EPr matrices, J.Red.Nat.But.Stds. 70B(1996), 47-77.
[3]. G.Marsaglia and G.P.H.Styan, Equalities and inequalities for ranks of matrices, Linear Multilinear Alg. 2(1974),269-292.
[4]. A.R.Meenakshi and S.Krishna moorthy, On q-k-EP matrices, Linear Alg.Appl. 269(1998), 219-232.
[5]. C.R.Rao and S.K.Mitra, Generalized Inverse of matrices and its applications, Wiley and sons, New York, 1971
[6]. A.Ben Israel and T.N.E.Greville, Generalized Inverses, Theory and applications, II edition, Canadian Math.Soc.Books in Mathematics, Springer Verlag, Network, Vol.15, 2003.
[7]. D.Carlson, E.Heynsworth and T.Markham, A generalization of the Schur complement by the Moore-penrose inverse, SIAM.J.Appl.Math., 26(1974), 169-175.
[8]. A.R.Meenakshi, On Schur complements in an EP matrix, Periadica Math.Hung., 16(1985),193-200
[9]. A.R.Meenakshi, Range symmetric matrices in Minkowski space, Bull.Malaysian Math.Sci.Soc.,23(2000), 45-52
[10]. A.R.Meenakshi and K.Bharathi, On k P Matrices in Minkowski space, Antarctica Journal of Mathematics, 8(3), (2011) 191-198
[11]. A.R.Meenakshi and K.Bharathi, On schur complement in k- EP matrices in Minkowski space, preprint.
[12]. A.R.Meenakshi and S.Krishna moorthy, On sums of k-EP matrices, Bull. Malaysian Math.Sci.Soc., 22(1999),117-126
[13]. K.Bharathi, Product of k-EP Block matrices in Minkowski space, Int.Nat. Journal of Math.Archive, Vol.5, No.1, 2014, 29-38.