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Study of Quantum Geometric Phase in Gravitomagnetism

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ABSTRACT: *We consider a situation where a collection of spin half charged particles circulate inside a rotating hollow sphere of massive body for finding probability distribution. The gravitomagnetic field of the sphere points along the Z-axis and it is being made to precess around the Z-axis maintaining a cone angle of θ. After a complete rotation the Berry's phase is, $\gamma_{(+)}(t) = -\pi\,(1-\cos\theta)$ and hence the probability distribution becomes modified due to accumulation of Berry's phase. This originates from the precessing gravitomagnetic field. In an experiment, there will apper a pattern which differs from that without the changing gravitomagnetic field of the rotating and precessing hollow sphere. The field of a rotating hollow sphere whose mass and rotation, (not charge) gives rise to the gravitomagnetic field because there is no Berry's phase due to gravitomaagnetic field which is used to make the beams to circulate in circles.*

Keywords: *Gravitomagnetic field, hollow sphere, and Berry's phase.*

I. INTRODUCTION

The Berry phase [1] has revolutionized quantum mechanics and perpetrated many areas of physics, including magnetism. It means that the wave function acquires a geometrical phase, $|\psi\rangle \rightarrow \exp(i\gamma)|\psi\rangle$ that is unrelated to the dynamical phase, $\exp(-Ht\,/\,\hbar)$, from the time-dependent Schrödinger equation. The Berry phase is important for the understanding of the orbital magnetic moment of itinerant electrons, for quantum entanglement, and for a number of magnetotransport phenomena, such as the anomalous Hall effect [2]. In fact, the Berry phase of a spin-1/2 particle in an adiabatically changing magnetic field is one of the first examples of this phenomenon [1]. The phase is well-defined when the field forms a closed loop and is essentially equal to the solid angle enclosed by the field, that is, to the corresponding area on the unit sphere. For spin-1/2 particles, the Berry phase obeys the simple relation [1, 6],

$$\gamma = \Omega/2,$$

where Ω is the solid angle of the loop. For nanoparticles that can be described as rigidly exchange-coupled macrospins of size, *S*, this equation changes to $= \Omega S$ [6].
The Berry phase is created by varying the field angle rather than the field strength, because it is an adiabatic process and the spin is always parallel to the external magnetic field. Any change in the magnitude of the field yields an ordinary Schrödinger-type dynamical phase, which does not interfere with the Berry phase [1].

In fact, the Berry phase of a spin in a magnetic field may actually be considered as a "zero-energy" effect [6].The external field, **H**(t), corresponds to a time-independent adiabatic energy, $E_0 = -\mu_0\mu_B\,\boldsymbol{\sigma}.\mathbf{H}$, so that one can use a simple unitary transformation to adjust the zero of the energy scale and ensure that $\mathrm{E}_0 = 0$. The nontrivial meaning of the corresponding "$H = 0$" problem is easily seen by considering the action, $S = \int \mathcal{L}\mathrm{dt}$, whose straight minimization corresponds to the system's classical motion, whereas the path integral, $\int exp\,(iS/\hbar)DxDp$, yields the quantum-mechanical amplitude. The Lagrangian, $\mathcal{L}$, contains not only the Hamiltonian, *H*, but also a geometrical contribution describing the phase space. For the linear motion of a particle, $\mathcal{L} = p\ \ \mathrm{dx/dt} - H$, however, the "flat" character of the p-x phase space makes the geometrical term uninteresting and the physics is determined by the Hamiltonian. However, the cross-product commutation rules for spins correspond to spin precession and mix the x, y, and z spin components, meaning that the geometrical term in L cannot be neglected [6].

The simplest way to rationalize the Berry phase is to assume a time-independent field magnitude, *H*. In the considered adiabatic limit, this corresponds to a constant Zeeman energy, $E = -\mu_0\mu_B\ H$, and the previously mentioned unitary transformation ensures that we can use $E = 0$. The dynamical phase is, therefore, zero and the Schrödinger equation predicts an unchanged wave function, $|\psi(t)> = |\psi(0)>$. However, this unchanged wave function is contradictory to our starting assumption that the field leads to an adiabatic rotation of the spin, $|\psi(t)> = |\psi(0,\varphi)>$. The paradox is solved by the Berry phase, which yields the correct wave function without changing the Hamiltonian.

In an addition, we consider a situation where a collection of spin half charged particles circulate inside a rotating hollow sphere of massive body. The gravitomagnetic field of the sphere points along the Z-axis (Fig. 1) and it is being made to precess around the Z-axis maintaining a cone angle of θ. This precession is slow enough to be applicable to the Berry's phase accumulation. After a complete rotation the Berry's phase is $\gamma_{(+)}(t) = -\pi\ (1-\cos\theta)$, and hence the probability distribution becomes modified due to accumulation of Berry's phase. In an experiment, there will appear a pattern which differs from that without the changing gravitomagnetic field of the rotating and precessing hollow sphere.

II. Indentations and Equations

We consider a situation where a collection of spin half charged particles circulate inside a rotating hollow sphere of massive body.

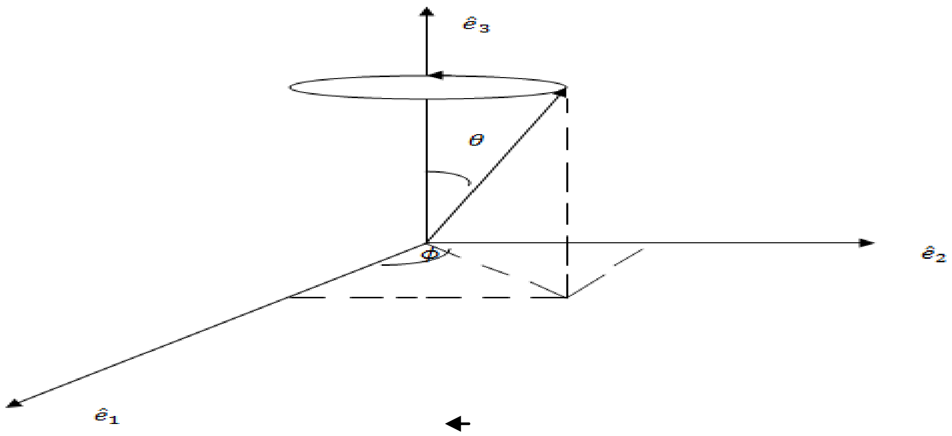

Fig.1. The gravitomagnetic field of the sphere points along the Z-axis and it is being made to precess around the Z-axis maintaining a cone angle of θ.This precession is slow enough to be applicable to the Berry's phase accumulation.

The gravitomagnetic field $\vec{B}$ is given by,

$$\vec{B} = B sin\theta cos\varphi\, \hat{e}_1 + B sin\theta sin\varphi\, \hat{e}_2 + B cos\theta\, \hat{e}_3 \tag{1}$$
$$= B[cos(\omega_1 t) sin\theta \hat{e}_1 + sin(\omega_1 t) sin\theta \hat{e}_2 + cos\theta \hat{e}_3]$$

$$= \alpha[cos(\omega_1 t) sin\theta\, \hat{e}_1 + sin(\omega_1 t) sin\theta\, \hat{e}_2 + cos\theta \hat{e}_3] \tag{2}$$

where $\phi = \omega_1 t,\ B \to \alpha\hat{n}, \alpha = \frac{4GM\omega}{3cR_0}$,

R_0 ≡radius of the rotating sphere and ω ≡rotational angular velocity about Z-axis of the rotating sphere.
The Hamiltonian of the interaction of the spin of the particles with the gravitomagnetic field is

$$H = \vec{S}\cdot\vec{B} \tag{3}$$

where, spin, $\vec{S} = \frac{1}{2}\vec{\sigma}$ and magnetic field, $\vec{B} = \alpha\hat{n}$.

Hence,

$$
\begin{aligned}
H &= \frac{1}{2}\vec{\sigma}\cdot\vec{B} \\
&= \frac{1}{2}\sigma_x B_x + \frac{1}{2}\sigma_y B_y + \frac{1}{2}\sigma_z B_z \\
&= \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}B_x + \frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}B_y + \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}B_z \\
&= \begin{pmatrix}0 & \frac{B_x}{2}\\ \frac{B_x}{2} & 0\end{pmatrix} + \begin{pmatrix}0 & -i\frac{B_y}{2}\\ i\frac{B_y}{2} & 0\end{pmatrix} + \begin{pmatrix}\frac{B_z}{2} & 0\\ 0 & -\frac{B_z}{2}\end{pmatrix} \\
&= \begin{pmatrix}0 & \frac{Bcos(\omega_1 t)sin\theta}{2}\\ \frac{Bcos(\omega_1 t)sin\theta}{2} & 0\end{pmatrix} + \begin{pmatrix}0 & -i\frac{Bsin(\omega_1 t)sin\theta}{2}\\ i\frac{Bsin(\omega_1 t)sin\theta}{2} & 0\end{pmatrix} + \begin{pmatrix}\frac{B\cos\theta}{2} & 0\\ 0 & -\frac{B\cos\theta}{2}\end{pmatrix} \\
&= \begin{pmatrix}\frac{1}{2}B\cos\theta & \frac{1}{2}Be^{-i\omega_1 t}sin\theta\\ \frac{1}{2}Be^{i\omega_1 t}sin\theta & -\frac{1}{2}B\cos\theta\end{pmatrix} \\
&= \begin{pmatrix}\beta\cos\theta & \beta e^{-i\omega_1 t}sin\theta\\ \beta e^{i\omega_1 t}sin\theta & -\beta\cos\theta\end{pmatrix}
\end{aligned}
$$

where $\beta = \frac{1}{2}B$. So,

$$
\begin{aligned}
H &= \frac{1}{2}\begin{pmatrix}B\cos\theta & Bcos(\omega_1 t)sin\theta - iBsin(\omega_1 t)sin\theta\\ Bcos(\omega_1 t)sin\theta + iBsin(\omega_1 t)sin\theta & -B\cos\theta\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}Z & X - iY\\ X + iY & -Z\end{pmatrix}
\end{aligned}
$$

where,

X= $Bcos(\omega_1 t)sin\theta$

Y= $Bsin(\omega_1 t)sin\theta$

Z= $B\cos\theta$

The eigenvalues are,

$$
\begin{aligned}
E_+ = -E_- &= \frac{1}{2}(X^2 + Y^2 + Z^2)^{\frac{1}{2}} \\
&= \frac{1}{2}[B^2cos^2(\omega_1 t)sin^2\theta + B^2sin^2(\omega_1 t)sin^2\theta + B^2cos^2\theta]^{\frac{1}{2}} \\
&= \frac{1}{2}B\ [sin^2\theta\{cos^2(\omega_1 t) + sin^2(\omega_1 t)\} + cos^2\theta]^{\frac{1}{2}} \\
&= \frac{1}{2}B\ [sin^2\theta + cos^2\theta]^{\frac{1}{2}} = \frac{1}{2}B
\end{aligned}
$$

So we can write,

$$E_+ = \frac{B}{2} = \frac{\alpha}{2} \tag{4}$$

and $$E_- = -\frac{B}{2} = -\frac{\alpha}{2} \tag{5}$$

We know ,original gravitomagnetic field,

$$\vec{B} = \hat{Z}\frac{4GM\omega}{3cR_0}$$

$$= \alpha \hat{Z} \tag{6}$$

When ,this $\vec{B}$ field precesses ,then $\vec{B} = \alpha \hat{n}$. Now, $(\vec{\sigma} \cdot \hat{n})^2 = 1$. Eigenvalues are, $\frac{\alpha}{2}, -\frac{\alpha}{2}$ and $\frac{\alpha}{2} = \beta$.

The eigenstates of $\beta \ \vec{\sigma} \cdot \hat{n}$ can be found as

$$\beta \ \vec{\sigma} \cdot \hat{n} \, \chi_n^{\pm} = \pm \beta \, \chi_n^{\pm} , \tag{7}$$

We know from (Ref.[46]),

$$\chi_n^{+} = \frac{1}{\sqrt{2(1+n_z)}} \{(1+n_z)\chi^{+} + (n_z + i n_y)\chi^{-}\}$$

$$= \frac{1}{\sqrt{2(1+\cos\theta)}} \{(1+\cos\theta)\chi^{+} + (\cos(\omega_1 t)\sin\theta + i\sin(\omega_1 t)\sin\theta)\chi^{-}\}$$

$$= \frac{1}{\sqrt{2(1+\cos\theta)}} \{(1+\cos\theta)\chi^{+} + \sin\theta \, e^{i\omega_1 t} \chi^{-}\}$$

$$= \frac{1}{\sqrt{2(1+\cos\theta)}} \left\{(1+\cos\theta)\begin{pmatrix}1\\0\end{pmatrix} + \sin\theta \, e^{i\omega_1 t}\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

$$\chi_n^{+} = \frac{1}{\sqrt{2(1+\cos\theta)}} \left[\begin{pmatrix}1+\cos\theta\\0\end{pmatrix} + \begin{pmatrix}0\\ \sin\theta \, e^{i\omega_1 t}\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2(1+\cos\theta)}} \begin{pmatrix}1+\cos\theta\\ \sin\theta \, e^{i\omega_1 t}\end{pmatrix} \tag{8}$$

Eigen state of E_{+} is

$$\psi_{(+)}(t) = \frac{1}{\sqrt{2(1+\cos\theta)}} \begin{pmatrix}1+\cos\theta\\ \sin\theta \, e^{i\omega_1 t}\end{pmatrix} \tag{9}$$

Final spin state at $t = t$,

$$\psi_{(+)}(t) = e^{iE_{(+)}t/\hbar} \, e^{i\gamma_{(+)}(t)} \, \psi_{(+)}(t) \tag{10}$$

where , $\gamma_{(+)}(t)$ is Berry's phase.

$$\gamma_{(+)}(t) = i \int_0^t < \psi_{(+)}(t')| \frac{\partial \psi_{(+)}(t')}{\partial t'} > dt' \tag{11}$$

Now,

$$H\psi_{(+)}(t') = \frac{1}{\sqrt{2(1+\cos\theta)}} \begin{pmatrix}\beta^2(1+\cos\theta)\\ \beta^2 \sin\theta \, e^{i\omega_1 t'}\end{pmatrix}$$

$$= \beta \frac{\beta}{\sqrt{2(1+\cos\theta)}} \begin{pmatrix}1+\cos\theta\\ \sin\theta \, e^{i\omega_1 t'}\end{pmatrix} \tag{12}$$

Then, $$\frac{\partial \psi_{(+)}(t')}{\partial t'} = \begin{pmatrix}0\\ \frac{\beta i \omega_1 \sin\theta \, e^{i\omega_1 t'}}{\sqrt{2(1+\cos\theta)}}\end{pmatrix} \tag{13}$$

$$< \psi_{(+)}(t')| \frac{\partial \psi_{(+)}(t')}{\partial t'} > = \left(\frac{\beta(1+\cos\theta)}{\sqrt{2(1+\cos\theta)}}, \quad \frac{\beta \sin\theta \, e^{-i\omega_1 t}}{\sqrt{2(1+\cos\theta)}}\right) \begin{pmatrix}0\\ \frac{\beta i \omega_1 \sin\theta \, e^{i\omega_1 t'}}{\sqrt{2(1+\cos\theta)}}\end{pmatrix}$$

$$= \frac{\beta^2 i \omega_1 \sin^2\theta}{2(1+\cos\theta)}$$

$$= \frac{i\omega_1}{2}(1-\cos\theta) \tag{14}$$

Finally, we find

$$\gamma_{(+)}(t) = i \int_0^t < \psi_{(+)}(t')| \frac{\partial \psi_{(+)}(t')}{\partial t'} > dt'$$

$$= i \int_0^t \frac{\omega_1}{2}(1-\cos\theta) dt'$$

$$= -\frac{\omega_1 t}{2}(1-\cos\theta)$$

$$= -\frac{1}{2}\Omega(C) \tag{15}$$

where, $\Omega(C) \equiv Solid\ angle$

$$= \omega_1 t (1-\cos\theta) \tag{16}$$

For a complete rotation, t$= \frac{2\pi}{\omega_1}$

$$\Omega(C) = \omega_1 \frac{2\pi}{\omega_1}(1-\cos\theta)$$
$$= 2\pi\,(1-\cos\theta) \quad (17)$$

$$\gamma_{(+)}(t) = -\frac{1}{2}2\pi\,(1-\cos\theta)$$
$$= -\,\pi\,(1-\cos\theta) \quad (18)$$

After a complete rotation, the wavefunction for a superposition of a beam of spin-$\frac{1}{2}$ particles in gravitomagnetic field with another beam but without gravitomagnetic field is,

$$\psi = \psi_0 + \psi_0 e^{i\gamma(c)}$$
$$= \psi_0\left(1+e^{i\gamma(c)}\right)$$
$$= \psi_0\,(1+\cos\gamma(c) + i\sin\gamma(c)) \quad (19)$$

Then, $$|\psi|^2 = |\psi_0|^2\{1+\cos\gamma(c)+i\sin\gamma(c)\}\{1+\cos\gamma(c)-i\sin\gamma(c)\}$$
$$= |\psi_0|^2\{(1+\cos\gamma(c))^2+\sin^2\gamma(c)\}$$
$$= |\psi_0|^2\{1+2\cos\gamma(c)+\cos^2\gamma(c)+\sin^2\gamma(c)\}$$
$$= |\psi_0|^2\{2+2\cos\gamma(c)\}$$
$$= |\psi_0|^2\{2(1+\cos\gamma(c))\}$$
$$= |\psi_0|^2 2\left(cos^2\frac{\gamma}{2}+sin^2\frac{\gamma}{2}+cos^2\frac{\gamma}{2}-sin^2\frac{\gamma}{2}\right)$$
$$= |\psi_0|^2 2\cdot 2\,cos^2\frac{\gamma}{2}$$
$$= 4|\psi_0|^2\,cos^2\frac{\gamma}{2} \quad (20)$$

Hence, the probability distribution becomes modified due to accumulation of Berry's phase .In an experiment, there will appear a pattern which differs from that without the changing gravitomagnetic field of the rotating and precessing hollow sphere.

III. Conclusion

Our problem is to find a probability distribution when we consider a situation where a collection of spin half charged particles circulate inside a rotating hollow sphere of massive body, and the gravitomagnetic field of the sphere points along the Z-axis and it is being made to precess around the Z-axis maintaining a cone angle of θ. Quantum geometric phase is analyzed to emphasize the Berry's phase derivation, and coupling gravitomagnetism-spin and Berry phase which helped us for solving our problem. we have examined a probability distribution which is modified due to a complete rotation or Berry's phase accumulation which originates from the precessing gravitomagnetic field. In an experiment, there will appear a pattern which differs from that without the changing gravitomagnetic field of the rotating and precessing hollow sphere.

Now we propose an experiment that the field of a rotating hollow sphere whose mass and rotation, (not charge) gives rise to the gravitomagnetic field because there is no Berry's phase due to electromagnetic field which is used to make the beams to circulate in circles. In conclusion, we have reviewed the phenomena of the probability distribution in the context of Berry's phase using the available literature. We hope that this paper will help other researchers in the field of gravitomagnetism.

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