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Fall Coloring And B-Coloring In Knight Graphs

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ABSTRACT: *A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has neighbor in all other color classes. Any such vertex is called as a colorful vertex. The b-chromatic numbers b(G) is the largest integer k. Such that G admits a b-coloring with k-colors. A fall coloring of a graph G is a proper coloring such that every vertex of G has neighbors in all the other color classes [3]. The b-spectrum $S_b(G)$ of G is defined by $S_b(G)=\{k\in N;\ \chi(G)\le K\le b(G)\}$ and G is b-colorable with K-colors}. A graph G is b-continuous if $S_b(G)=\{\chi(G), b(G)\}$. In this paper we obtain infinite number of Knight graphs $KN_{3,K}$, $KN_{4,K}$ and $KN_{5,K}$ which satisfier $\chi(G)\le\varphi(G)\le\Delta(G)+1$. Also we prove that these graphs are b-continuous for some value of k.*

Keywords :- b-coloring, b-continuous, knight graphs

I. INTRODUCTION

A k-vertex coloring of a graph G is a assignment of k-colors 1,2,…, k to the vertices. The coloring is proper if no two district adjacent vertices share the same color. Many graph invariants related to coloring have been defined most of them try to minimize the number of colors used to color the vertices under some constraints for some other invariants it is meaningful to try maximize this number. The b-chromatic number is such an example [6]. Corresponding to the chess pieces queen, rook, bishop, knight and king there are graphs $Q_{j,k}$, $R_{j,k}$, $KN_{j,k}$ and $KI_{j,k}$ each of order n=jk where the vertex set corresponds to the jk squares of j-by-k chess board and two vertices are adjacent if and only if the given chess piece can go from one of the two corresponding squares to the other corresponding squares in one move [1] Kratochvil et al. showed that for a d-regular graph G with atleast d^4 vertices $\varphi(G)=d+1$[2].

We, let $v_{i,j}$ or simply (i, j) denote the vertex in the i^{th} row and j^{th} coloumn.

$KN_{3,4}$ the vertex $v_{2,1}$ = (2, 1) has the closed neighborhood N[(2, 1)] = {(2,1), (1,3), (3,3)} with cardinality |N[(2,1)]| = 3 and the vertex $v_{3,2}$=(3,2) has the closed neighborhood N[(3,2)]={(3,2), (1,1), (1,3), (2,4)} with cardinality |N|(3,2)] = 4.

For an excellent survey about Knight graphs,
see Hedetniemi and Reynolds [4]. Another wonderful exposition concerning the parameters is given by Watkins in [5].
Here we give an example of a Knight graph on 16 vertices.

The knight graphs $KN_{3,k}$

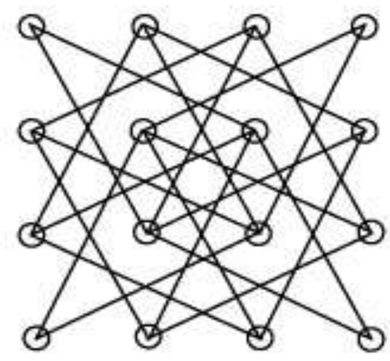


Fig.1:1:1

The knight graphs $KN_{3,k}$

We study the b-coloring number of all Knight graphs KN3,k for k≥4; $KN_{4,K}$ K≥7, $KN_{5,K}$ K≥4. Here we give an example of a knight graphs on 16 vertices.

Lemma 1:2:1

The graph $KN_{3,4}$ is fall colorable with 2 and 3 colors. Also G is b-continuous and φ(G)=3.

Proof

Let the vertex set of $G=KN_{3,4}$ be V(G)={(a,b): 1 ≤ a ≤ 3 and 1 ≤ b ≤4}. The next figure illustrates the fall coloring of $KN_{3,4}$ with 2 and 3 colors.

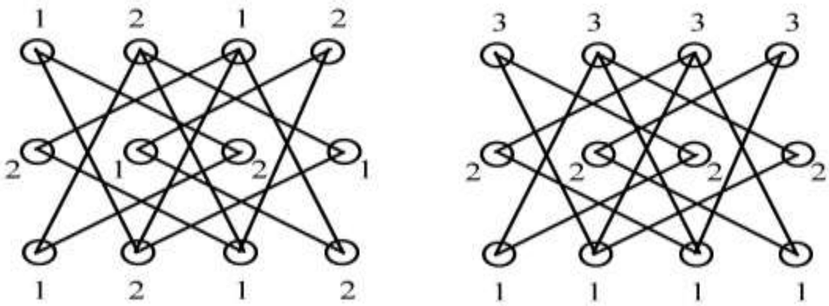


Fig.1.2:2

Since δ(G)= 2 and Δ(G)=3, it is easy to observe the following: The fall chromatic number of G is 2; the fall achromatic number of G is 3. To prove that the b-coloring number φ(G)=3 and G is b-continuous, it remains to show that G is not b-colorable with 4 colors. On the contrary, assume that G is b-colorable with 4 colors. Since G has only four vertices namely, (1,2), (1,3), (3,2) and (3,3) of degree 3 and G is b-colorable with 4 colors, we can conclude that these four vertices should be colorful.

Without loss of generality we can assume Fig.1:2:2a. Note that the colorful vertices are marked with dark circles. Consider the colorful vertex (1,2) with color 1. It is already adjacent with the vertex (3,3) which has been colored with colo3 3. Hence the other two neighbors of (1,2), namely (3,1) and (2,4) should receive colors 2 and 4.

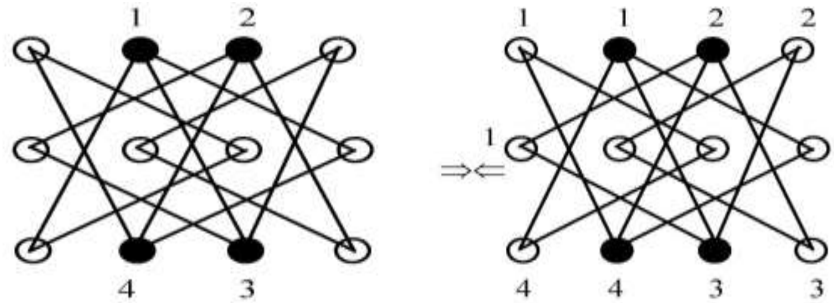


Fig. 1:2:2a **Fig. 1:2:2b**

Since (2,4) is adjacent with color 4, we should color the vertex (3,1) by color 4 as (as shown in Figure 1.2.1b).

Similarly, we can prove that c(1,1) =1, c((1,4))=2 and c((3,4))=3 (as shown in Fig. 1.2.1b). Now to make the vertex (1,3) colorful, we should color the vertex (2,1) with color 1 and which will contradict the fact that (3,3) is a colorful vertex with color 3. Hence G is not b-colorable with 4 colors.

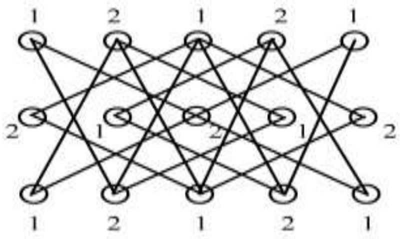
Lemma 1:2:2

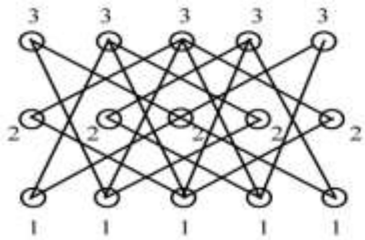
The graph G=K $N_{3,5}$ is b-continuous and the b-chromatic number φ(G)=4.

Proof

Let the vertex set of $G=KN_{3,5}$ be V(G)={(a, b): 1≤ a ≤ 3 and 1 ≤ b ≤ 5}.

The next figure illustrates the fall coloring of $KN_{3,5}$ with 2 and 3 colors. Note that in these two colorings, all the vertices are colorful.





The next figure illustrates the fall coloring of $KN_{3,4}$ with 4 colors and the colorful vertices are marked with dark circle. Since there are only 3 vertices are marked with dark circles. Since there are only 3 vertices of degree 4, G is not b-colorable with 5 colors.

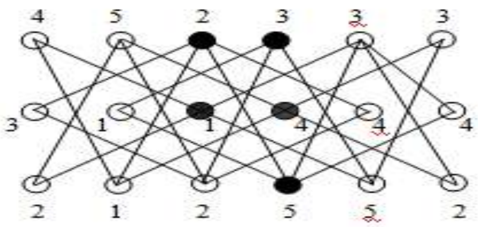


Fig.1:2:4

Since $\delta(G)=2$ and $\Delta(G)=4$, from the above discussion, we can observe the following: The fall chromatic number of G is 2; the fall achromatic number of G is 3; the b-coloring number $\varphi(G)=4$ and G is b-continuous. □

Lemma 1:2:3

The graph $G = KN_{3,6}$ is b-continuous and the b-chromatic number $\varphi(G)=5$.

Proof

Let the vertex set of $G=KN_{3,6}$ be $V(G)=\{(a, b): 1 \leq a \leq 3 \text{ and } 1 \leq b \leq 6\}$.

As in the proof of above we can prove easily that the graph G is b-colorable with 2 colors, 3 colors and 4 colors. The fig.1.2.5 illustrate the b-coloring of G with 5 colors.

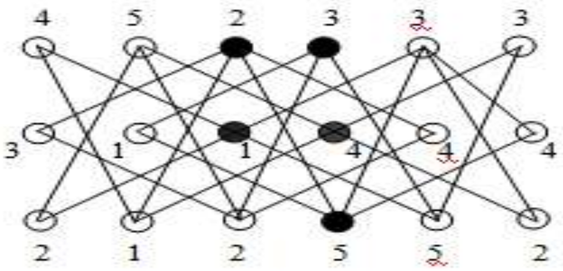


Fig.1:2:5 with 5 colors

Since $\delta(G)=2$ and $\Delta(G)=4$, from the above discussion, we can observe the following: The fall chromatic number of G is 2; the fall achromatic number of G is 3; the b-coloring number $\varphi(G) =5$ and G is b-continuous.

Lemma 1:2:4

Let $k \geq 7$ be an integer and $G=K N_{3,k}$. Then the graph G is b-continuous and $\varphi(G)=5$.

Proof

Let the vertex set of $G=KN_{3,k}$ be $V(G)=\{(a, b) : 1 \leq a \leq 3 \text{ and } 1 \leq b \leq k\}$.

As in the proof of Lemma 1.2.1, we can prove easily that the graph G is b-colorable with 2 colors, 3 colors and 4 colors. Fig. 1.2.6 illustrates the b-coloring of G with 4 colors. The colorful vertices are marked with dark circles.

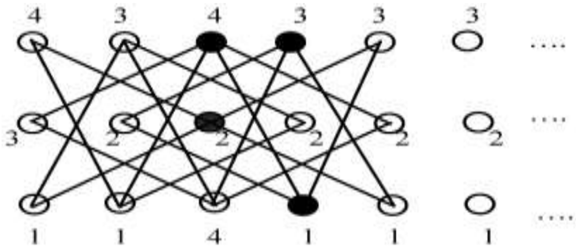


Fig.1:2:6

Note that the remaining vertices (not shown in figure) in the first row are colored with color 3, the remaining vertices in the second row are colored with color 2, and the vertices in the third row are colored with color 1.

Fig.1:2:7 illustrates the b-coloring of G with 5 colors.

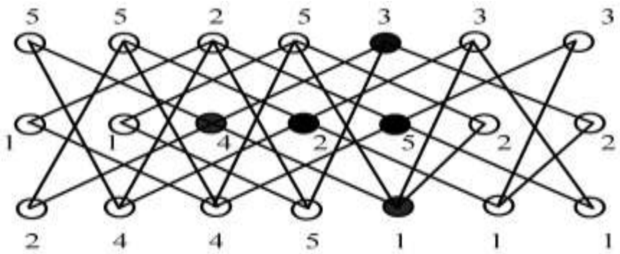


Fig.1:2:7 b-coloring of $KN_{3,k}$ with 5 colors

Note that the remaining vertices in the first row are colored with color 3; the remaining vertices in the second row are colored with color 2; and the vertices in the third row are colored with color 1.
Since $\Delta(G)=4$, we have $\varphi(G) \leq 1 + \Delta(G)=5$. Since $\delta(G)=2$, it is easy to say that $\chi(G)\geq 2$ and hence $[\chi(G), \varphi(G)]=[2,5]$. We proved that G is b-colorable with i colors for each i with $2\leq i\leq 5$. Hence G is b-continuous. □

We can conclude the following result by using Lemma, 1.2.1, Lemma 1.2.2,
Lemma 1.2.3 and Lemma 1.2.4.

Theorem 1:2:5

The graph $G = KN_{3,k}$ is b-continuous with $\varphi(G)= \Delta(G)+1$ if and only if, $k \geq 5$.
The Knight graphs $KN_{3,k}$.

1:3 The knight Graphs $KN_{4,k}$

In this section, we prove that the Knight graphs $KN_{4,k}$ are b-continuous for $k \geq 7$.

Further, we obtained a necessary and sufficient condition for the existence of b-coloring with $\Delta(G)+1$ colors in the graphs $KN_{4,k}$.

Lemma 1.3.1.

For $k \geq 8$, the graph $G=KN_{4,k}$ is fall colorable with 2 and 3 colors. Also G is b-continuous and $\varphi(G)=7$.

Proof

Let the vertex set of $G=KN_{4,k}$ be $V(G)=\{(a, b): 1 \leq a \leq 4 \text{ and } 1 \leq b \leq k\}$.

We call the vertices $\{(1,b): 1 \leq b \leq k\}$ first row vertices the vertices $\{(2, b)\ 1 \leq b\leq k\}$ as second row vertices, the vertices $\{(3, b): 1 \leq b \leq k\}$ as third row vertices, the vertices $\{(4, b): 1 \leq b \leq k\}$ as last or fourth row vertices.

If we color the first, third row vertices by color 1; and second, last row vertices by color 2, then the graph G is fall colorable with 2 colors.

If we color the first and last row vertices by color 1; second row vertices by color 2; and the third row vertices by color 3, then the graph G is fall colorable with 3 colors.

As in the proof of Lemma 1.2.4, we can prove that G is b-colorable with 4 and 5 colors.

The following figure illustrates the fall coloring of $KN_{4,k}$ with 6 colors. In the coloring, the rest of the vertices in the i^{th} row are colored by color i, for $1 \leq i \leq 4$.

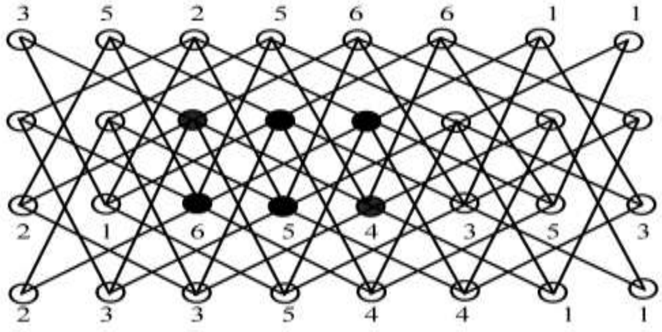


Fig.1:3:1

The following figures illustrate the fall coloring of $KN_{4,k}$ with 7 colors. In this coloring, the rest of the vertices in the first and last rows are colored by color 1; the second row vertices are colored by color 2; the third row vertices are colored by color 7.

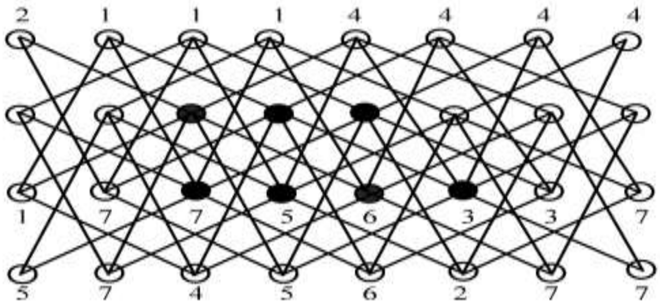


Fig.1:3:2 b-coloring of $KN_{4,K}$ with 7 colors

We proved the G is b-colorable with i colors for $2 \leq i \leq 7$. Since $\delta(G)=2$ and $\Delta(G)=6$, from the above discussion, we can observe the following: The fall chromatic number of G is 2; the fall achromatic number of G is 3; the b-coloring number $\varphi(G)=7$ and G is b-continuous.

Lemma 1.3.2

The graph $G=KN_{4,7}$ is b-continuous and the b-chromatic number $\varphi(G)=6$.

Proof

Note that $\Delta(G)=6$. Hence $\varphi(G) \leq 7$. But G has exactly 6 vertices of degree 6 and hence G is not b-colorable with 7 colors. As in the proof of above lemma, we can prove that G is b-colorable with i colors for each i with $2 \leq i \leq 6$.

Hence G is b-continuous and $\varphi(G) \leq 6$. □

By using Lemma 1.3.1 and 1.3.2 we can get the following result.

Theorem 1.3.3

The graph $G=KN_{4,k}$ is b-continuous with $\varphi(G)= \Delta(G)+1$ if, and only if, $k \geq 8$.

1.4 The Knight graphs $KN_{5,k}$

In this section, we prove that all the Knight graphs $KN_{5,k}$ $(k \geq 4)$ are b-continuous. Also we characterize Knight graphs $KN_{5,k}$ with $\varphi(KN_{5,k})=\Delta+1$.

Lemma 1.4.1

For $k \geq 4$, the graph $G=KN_{5,k}$ is fall colorable with 2 and 3 colors.

Proof:

Let the vertex set of $G=K N_{5,k}$ be $V(G)=\{(a, b): 1\leq a \leq 5 \text{ and } 1 \leq b \leq k\}$.

Note that, we call the vertices {(1, b): 1≤b≤k} as first row vertices, the vertices ((2, b):1≤b≤K} as second row vertices, the vertices {(3, b): 1≤b≤k} as third row vertices, the vertices {(4,b): 1≤b≤k} as fourth row vertices, and the vertices {(5,b): 1≤b≤k} as fifth or last row vertices.

We color the first, third and fifth row vertices using the color sequence 1,2,1,2…; second and fourth row vertices by the color sequence2, 1,2,1,…. Then the graph G is fall colorable with 2 colors. If we color the first and fourth row vertices by the color 1; second and fifty row vertices by color 2; and the third row vertices by color 3, then the graph G is fall colorable with 3 colors. □

Lemma 1.4.2

For k ≥ 4, the graph G = $KN_{5,k}$ is b-colorable with four and five colors and $\varphi(KN_{5,4})=5$.

Proof

The following figure illustrates the b-coloring of $KN_{5,k}$ (k≥4) with four colors.

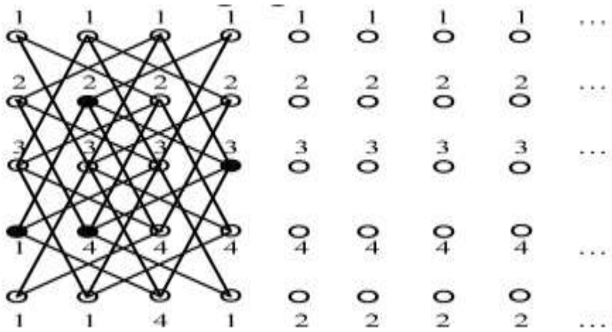


Fig.1:4:1 b-coloring of $KN_{5,k}$ with 4 colors

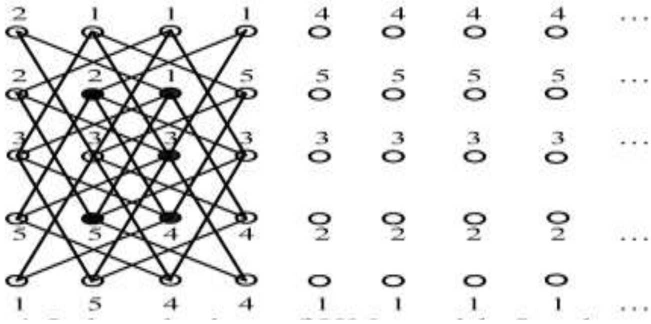


Fig.1:4:2 b-coloring of $KN_{5,k}$ with 5 colors

The above figure illustrates the b-coloring of $KN_{5,k}$, k≥4 with five colors.
Note that, $KN_{5,4}$ has only two vertices (3,2) and (3,3) of degree greater than or equal to 5 and hence $KN_{5,4}$ is not b-colorable with 6 colors. Thus $\varphi(KN_{5,4})=5$. □

Lemma 1.4.3

$\varphi(KN_{5,5})=5$

Proof

As proved in the Lemma 1.4.2, $KN_{5,5}$ is b-colorable with 2,3,4 and 5 colors.
Since K $N_{5,5}$ has only five vertices (2,3), (3,2), (3,3), (3,4) and (4,3), of degree greater than or equal to 5, $KN_{5,5}$ is not b-colorable with 6 colors. Hence $\varphi(KN_{5,5})=5$. □

Lemma 1.4.4

For k ≥ 6 the graph G=$KN_{5,k}$ is b-colorable with six colors. Also $\varphi(KN_{5,6})=6$.

Proof

As proved in the Lemma 1.4.2, $KN_{5,6}$ is b-colorable with 2,3,4 and 5 colors. The next figure illustrates the b-coloring of $KN_{5,k}$, k≥6 with six colors.
Note that $KN_{5,6}$ has only eight vertices (2,3), (2,4), (3,2), (3,3), (3,4), (3,5), (4,3) and (4,4), of degree greater than or equal to 6. Suppose $KN_{5,6}$ is b-colorable with 7 colors. Then there are two cases.

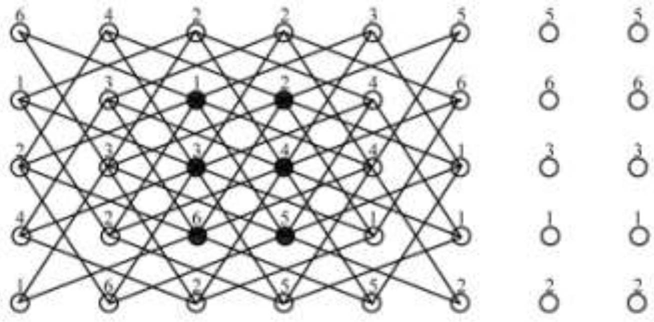


Fig.1:4:3

Case

The 7 colorful vertices are as given below

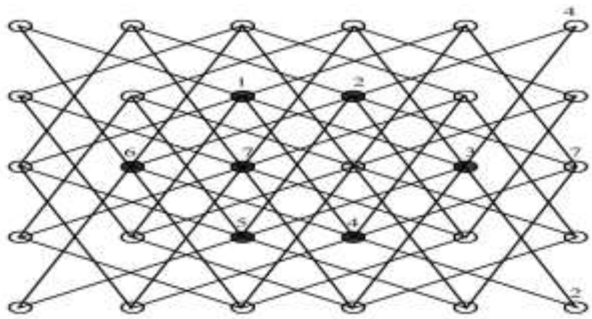


Fig.1:4:4

Note that if a vertex v has degree 6, then all its adjacent vertices receive different colors; otherwise v cannot be colorful. Consider the two vertices (2,4) and (4,4) with colors 2 and 4 respectively. To make these vertices colorful, we can color the vertex (3,6) either with color 3 or with color 7.

If c((3,6))=3, then to make the two vertices (2,4) and (4,4) colorful, we should have c((1,6))=c((5,6))=7.

Hence the vertex (3,5) with color 3 could not be colorful. Hence c((3,6))=7. Now to make the vertices (2,4) and (3,5) colorful, we should have c((1,6))=4 and to make the vertices (3,5) and (4,4) colorful, we should have c((5,6))=2 as shown in fig.1:4:4

In this case the vertex (3,5) with color 3 cannot have adjacency with color 7 and hence it is not colorful.

II. CASE

The 7 colorful vertices are as given below:

Consider the vertex (3,5) with color 3. To make this vertex colorful, we must have c((1,6))=7 or c((5,6))=7. Without loss of generality, assume that c((1,6))=7. (as given the above figure) In this case, without affecting the colorfulness of the vertices (2,4) and (3,5) with colors 2 and 3 respectively, the vertex (4,4) cannot have adjacency with color 7. Thus the vertex (4,4) with color 4 is not colorful a contradiction.

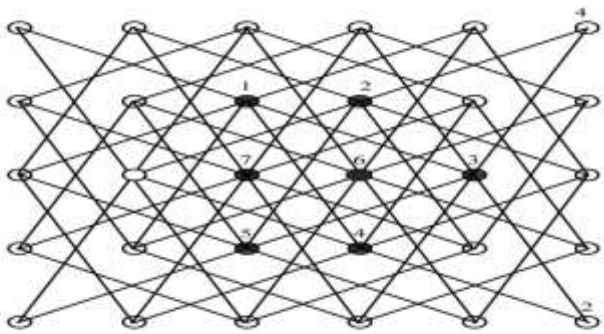


Fig.1:4:5

Similarly when c((5,6)) = 7, we will get contradiction. Hence $N_{5,6}$ is not b-colorable with 7 colors and so $\varphi(KN_{5,6})=6$. □

Lemma 1.4.5

For k ≥ 7,the graph G=$KN_{5,k}$ is b-colorable with 7 colors. Also φ($KN_{5,7}$)=7.

Proof

As proved in Lemma 1.4.2 and Lemma 1.4.4, $KN_{5,k}$ is b-colorable with 2,3,4,5 and 6 colors, when k ≥ 7. The next figure illustrate the b-coloring of K $N_{5,k}$, k ≥ 7 with seven colors.

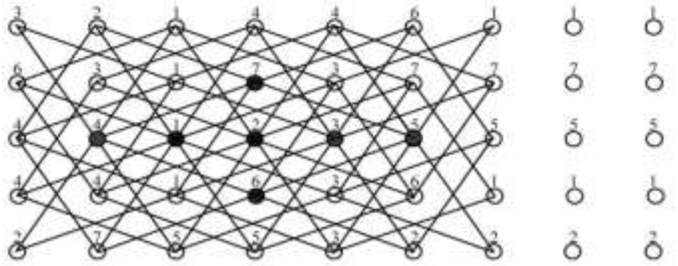


Fig.1:4:6

Since $KN_{5,7}$ has only three vertices (3,3), (3,4) and (3,5), of degree greater than or equal to 7, $KN_{5,7}$ is not b-colorable with 8 colors. Hence φ($KN_{5,7}$)=7.

Remark 1.4.6

Let G=$KN_{5,k}$ and k be an integer such that 8 ≤ k ≤ 11. Then G have less than 8 vertices of degree greater than or equal to 7.

Hence G is not b-colorable with 8 colors and hence φ(G)=7.

Lemma 1.4.7

For k≥12 the graph G=$KN_{5,k}$ is b-colorable with 8 colors. Also φ($N_{5,12}$)=8

Proof

As in the proof of Lemma 1.4.4, Lemma 1.5.5 and Lemma 1.4.5, $KN_{5,k}$ is b-colorable with 2,3,4,5,6 and 7 colors, when k ≥ 12. The next figure illustrates the b-coloring of $KN_{5,k}$, k≥12 with 8 colors.

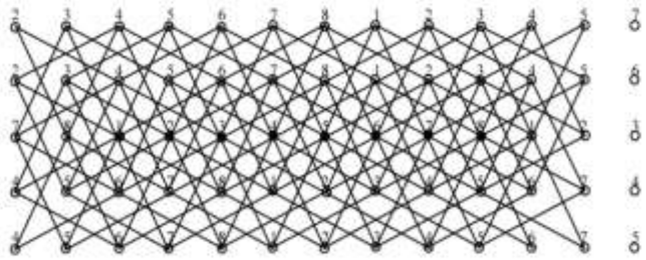


Fig.1:4:7

Note that $KN_{5,12}$ have only eight vertices of degree greater than or equal to 8. Hence $KN_{5,12}$ is not b-colorable with 9 colors and hence φ($KN_{5,12}$)=8. □

Lemma 1.4.8

For k≥13, the graph G=$KN_{5,k}$ is b-colorable with 9 colors. Also φ(G)=9.

Proof

As proved in the Lemma 1.4.2, Lemma 1.4.4, Lemma 1.4.5 and Lemma 1.4.7, $KN_{5,k}$ is b-colorable with 2,3,…,8 colors, when k ≥ 13. The b-coloring of $KN_{5,k}$, (k ≥ 13) with 9 colors.
Since Δ(G)=8, we have φ(G)= Δ(G)+1. □

From all the results in this section, we have the following theorem.

Theorem 1.4.9

a) The graphs $KN_{5,k}$, (k≥4) are b-continuous.
b) φ($KN_{5,4}$) = φ(K $N_{5,5}$) = 5.

c) $\varphi(K\ N_{5,6}) = 6$.
d) $\varphi(K\ N_{5,k}) = 7$, if $7 \leq k \leq 11$.
e) $\varphi(K\ N_{5,12}) = 8$
f) $\varphi(K\ N_{5,k}) = \Delta + 1$ if and only if $k \geq 13$.

III. CONCLUSION

This work may be extended in the following direction.

1. The graph $KN_{6,k}$ is b-continuous for all $k \geq 4$.
2. The graph $G = KN_{6,k}$ is b-continuous with $\varphi(G)=9$ if, and only if, $k \geq 7$.

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