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Common Fixed Point Theorem Satisfying Rational Contractive Condition on $T-$orbitally Complete Metric Space

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***ABSTRACT**: In this article we prove a common fixed point result satisfying rational contractive condition in T-orbitally complete metric space. We also give some corollaries which is equivalent to the proved theorem.*

***Keywords:** T-orbitally complete metric space, weakly compatible, generalized weakly contractive maps*

I. INTRODUCTION

In 1968, Kannan [1] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. A number of these papers dealt with fixed points for more than one map. In some cases commutativity between the maps was required in order to obtain a common fixed point. Sessa [2] coined the term weakly commuting. Jungck [3] generalized the notion of weak commutativity by introducing the concept of compatible maps and then weakly compatible maps [4]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. Also, during this time a number of researchers established fixed point theorems for pair of maps. One of the most popular generalizations of metric space is $T-$orbitally metric space. Our aim of this article is to obtain some common fixed point in $T-$orbitally metric space satisfying different rational contractive conditions. In 1997 Alber and Guerre- Delabriere [5] introduced the concept of weakly contractive map in Hilbert space and proved the existence of fixed point results. Rhoades [6] extended this concept in Banach space and established the existence of fixed points.

II. PRELIMINARIES

Definition 2.1 : For any $x_0 \in X$; $O(x_0) = \{T^n x_o ; n = 0,1,2,3 \ldots\ldots\}$ is said to the orbit of x_0 where, $T^0 = I$, is the identity map of X. $\overline{O(x_0)}$ represent the closer of $O(x_0)$.

A metric space X is said to be $T-$orbitally complete; if every Cauchy sequence Which is contained in $O(x)$ for all $x \in X$ converges to the point of X.

Here we note that every complete metric space is $T-$ orbitally complete for any T, but converges is not true.

Definition 2.2 : Let A and S be the mapping from a metric space X into itself, then the mapping is said to weakly compatible if they are commute at their coincidence points, that is,
$Ax = Sx$ implies that $ASx = SAx$.

Definition 2.3 A self map $T: X \to X$ is said to be generalized weakly contractive map if there exists a $\psi \in \Phi$ such that,

$$d(Tx, Ty) \le d(x, y) - \psi\big(d(x, y)\big)$$

with $\lim_{t\to\infty} \psi(t) = 0$ for all $x, y \in X$.

II. MAIN RESULT

Theorem 3.1 Let (X, d) be a $T-$ orbitally complete metric space, if A, B, S, T be the self mapping of X into itself such that;

3.1(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X.

3.1(ii) The pair (A, S) and (B, T) are weakly compatible and generalized weakly contractive map.

3.1(iii) for all $x, y \in \overline{O(x_0)}$ and $k \in [0,1)$, we define,

$$d(Ax, By) \le k.M(x, y) - \psi(M(x, y))$$

Where , $M(Ax,By) = \max\left\{\begin{matrix} \frac{d^2(Ax,Sx)+d^2(By,Ty)}{1+d(Sx,Ty)}, \frac{d^2(Sx,By)+d^2(Ax,Ty)}{1+d(Sx,Ty)}, \\ \frac{d(Ax,Sx).d(By,Ty)}{1+d(Sx,Ty)}, \frac{d(Sx,By).d(Ax,Ty)}{1+d(Sx,Ty)}, d(Sx,Ty) \end{matrix}\right\}$.

Then A, B, S, T have unique fixed point in $\overline{O(x_0)}$.

Proof:- We suppose that, $x_0 \in X$ arbitrary and we choose a poin t $x \in X$ such that,

$$y_0 = Ax_0 = Tx_1 \text{ and } y_1 = Bx_1 = Sx_2$$

In general there exists a sequence,

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

for $n = 1,2,3 \dots\dots\dots$

first we claim that the sequence $\{y_n\}$ is a Cauchy sequence for this from 3.1(iii) we have,

$$d(y_{2n},y_{2n+1}) \le k.M(Ax_{2n},Bx_{2n+1}) - \psi(M(Ax_{2n},Bx_{2n+1}))$$

$$d(y_{2n},y_{2n+1}) \le k\max\left\{\begin{matrix} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(Bx_{2n+1},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})}, \\ \frac{d^2(Sx_{2n},Bx_{2n+1})+d^2(Ax_{2n},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})} \\ \frac{d(Ax_{2n},Sx_{2n}).d(Bx_{2n+1},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})}, \\ \frac{d(Sx_{2n},Bx_{2n+1}).d(Ax_{2n},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})}, \\ d(Sx_{2n},Tx_{2n+1}), \end{matrix}\right\}$$

$$-\psi\left(\max\left\{\begin{matrix} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(Bx_{2n+1},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})}, \\ \frac{d^2(Sx_{2n},Bx_{2n+1})+d^2(Ax_{2n},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})} \\ \frac{d(Ax_{2n},Sx_{2n}).d(Bx_{2n+1},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})}, \\ \frac{d(Sx_{2n},Bx_{2n+1}).d(Ax_{2n},Tx_{2n+1})}{1+d(Sx_{2n},Tx_{2n+1})}, \\ d(Sx_{2n},Tx_{2n+1}), \end{matrix}\right\}\right)$$

$$d(y_{2n},y_{2n+1}) \le k\max\left\{\begin{matrix} \frac{d^2(y_{2n},y_{2n-1})+d^2(y_{2n+1},y_{2n})}{1+d(y_{2n-1},y_{2n})}, \frac{d^2(y_{2n-1},y_{2n+1})+d^2(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n})} \\ \frac{d(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}{1+d(y_{2n-1},y_{2n})}, \frac{d(y_{2n-1},y_{2n+1}).d(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n})}, \\ d(y_{2n-1},y_{2n}), \end{matrix}\right\}$$

$$-\psi\left(\max\left\{\begin{matrix} \frac{d^2(y_{2n},y_{2n-1})+d^2(y_{2n+1},y_{2n})}{1+d(y_{2n-1},y_{2n})}, \frac{d^2(y_{2n-1},y_{2n+1})+d^2(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n})} \\ \frac{d(y_{2n},y_{2n-1}).d(y_{2n+1},y_{2n})}{1+d(y_{2n-1},y_{2n})}, \frac{d(y_{2n-1},y_{2n+1}).d(y_{2n},y_{2n})}{1+d(y_{2n-1},y_{2n})}, \\ d(y_{2n-1},y_{2n}), \end{matrix}\right\}\right)$$

$$d(y_{2n},y_{2n+1}) \le k\max\left\{\begin{matrix} d(y_{2n-1},y_{2n}), d(y_{2n+1},y_{2n}), \\ d(y_{2n+1},y_{2n}), 0, d(y_{2n-1},y_{2n}) \end{matrix}\right\}$$
$$-\psi\left(\max\left\{\begin{matrix} d(y_{2n-1},y_{2n}), d(y_{2n+1},y_{2n}), \\ d(y_{2n+1},y_{2n}), 0, d(y_{2n-1},y_{2n}) \end{matrix}\right\}\right)$$

There arise three cases:

Case- 1 If we take

$$\max\left\{\begin{matrix} d(y_{2n-1},y_{2n}), d(y_{2n+1},y_{2n}), \\ d(y_{2n+1},y_{2n}), 0, d(y_{2n-1},y_{2n}) \end{matrix}\right\} = d(y_{2n-1},y_{2n})$$

Then we have

$$d(y_{2n},y_{2n+1}) \le k.d(y_{2n-1},y_{2n}) - \psi(d(y_{2n-1},y_{2n}))$$

Taking limit then we have, $\lim_{n\to\infty}\psi(d(y_{2n-1},y_{2n})) \to 0$ and hence

$$d(y_{2n},y_{2n+1}) \le k.d(y_{2n-1},y_{2n}).$$

Case- 2 If we take

$$\max\left\{\begin{matrix} d(y_{2n-1},y_{2n}), d(y_{2n+1},y_{2n}), \\ d(y_{2n+1},y_{2n}), 0, d(y_{2n-1},y_{2n}) \end{matrix}\right\} = d(y_{2n+1},y_{2n})$$

then we have

$$d(y_{2n},y_{2n+1}) \le k.d(y_{2n+1},y_{2n}) - \psi(d(y_{2n+1},y_{2n}))$$

taking limit then we have, $lim_{n\to\infty}\psi\big(d(y_{2n-1},y_{2n})\big)\to 0$ and hence

$$d(y_{2n},y_{2n+1}) \le k.d(y_{2n+1},y_{2n})$$

which contradiction.

Case- 3 If we take

$$max\begin{Bmatrix} d(y_{2n-1},y_{2n}),d(y_{2n+1},y_{2n}),\\ d(y_{2n+1},y_{2n}),0,d(y_{2n-1},y_{2n})\end{Bmatrix} = 0$$

then we have

$$d(y_{2n},y_{2n+1}) \le 0$$

which contradiction.

from the above all three cases we have

$$d(y_{2n},y_{2n+1}) \le k.d(y_{2n-1},y_{2n})$$

processing the same way we have

$$d(y_{2n},y_{2n+1}) \le k^{2n}.d(y_0,y_1)$$

or

$$d(y_n,y_{n+1}) \le k^n.d(y_0,y_1)$$

for any $m > n$ we have

$$d(y_n,y_m) \le d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + \cdots \ldots + d(y_{m-1},y_m)$$
$$d(y_n,y_m) \le (k^n + k^{n+1} + \ldots\ldots + k^{m-1})d(y_0,y_1)$$
$$d(y_n,y_m) \le \frac{k}{1-k}\, d(y_0,y_1).$$

As $n\to\infty$, it follows that $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. That is we can write;

$$lim_{n\to\infty} y_n = lim_{n\to\infty} Ax_{2n} = lim_{n\to\infty} Tx_{2n+1}$$
$$= lim_{n\to\infty} Bx_{2n+1} = lim_{n\to\infty} Sx_{2n+2} = y.$$

Now let $T(X)$ is closed subset of X such that, $Tv = y$.

We prove that $Bv = y$ for this again from 3.1(iii),

$$(Ax_{2n},Bv) \le k\ max\begin{Bmatrix} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(Bv,Tv)}{1+d(Sx_{2n},Tv)}, \frac{d^2(Sx_{2n},Bv)+d^2(Ax_{2n},Tv)}{1+d(Sx_{2n},Tv)},\\ \frac{d(Ax_{2n},Sx_{2n})\,d(Bv,Tv)}{1+d(Sx_{2n},Tv)}, \frac{d(Sx_{2n},Bv)d(Ax_{2n},Tv)}{1+d(Sx_{2n},Tv)},\\ d(Sx_{2n},Tv)\end{Bmatrix}$$
$$-\psi\left(max\begin{Bmatrix} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(Bv,Tv)}{1+d(Sx_{2n},Tv)}, \frac{d^2(Sx_{2n},Bv)+d^2(Ax_{2n},Tv)}{1+d(Sx_{2n},Tv)},\\ \frac{d(Ax_{2n},Sx_{2n})\,d(Bv,Tv)}{1+d(Sx_{2n},Tv)}, \frac{d(Sx_{2n},Bv)d(Ax_{2n},Tv)}{1+d(Sx_{2n},Tv)},\\ d(Sx_{2n},Tv)\end{Bmatrix}\right)$$

$$d(y,Bv) \le k\ max\{d(y,Tv),d(y,y),d(Bv,y),d(y,Bv),0\}$$
$$-\psi(max\{d(y,Tv),d(y,y),d(Bv,y),d(y,Bv),0\})$$
$$d(y,Bv) < k\,.d(y,Bv)$$

Which contradiction,

Hence $Bv = y = Tv$ *and that* $BTv = TBv$ *implies that* $By = Ty$.

Now we proof that $By = y$ for this again from 3.1(iii)

$$d(Ax_{2n},By) \le k\ max\begin{Bmatrix} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(By,Ty)}{1+d(Sx_{2n},Ty)}, \frac{d^2(Sx_{2n},By)+d^2(Ax_{2n},Ty)}{1+d(Sx_{2n},Ty)}\\ \frac{d(Sx_{2n},By).d(Ax_{2n},Ty)}{1+d(Sx_{2n},Ty)}, \frac{d(Ax_{2n},Sx_{2n}).d(By,Ty)}{1+d(Sx_{2n},Ty)},\\ d(Sx_{2n},Ty)\end{Bmatrix}$$
$$-\psi\left(max\begin{Bmatrix} \frac{d^2(Ax_{2n},Sx_{2n})+d^2(By,Ty)}{1+d(Sx_{2n},Ty)}, \frac{d^2(Sx_{2n},By)+d^2(Ax_{2n},Ty)}{1+d(Sx_{2n},Ty)}\\ \frac{d(Sx_{2n},By).d(Ax_{2n},Ty)}{1+d(Sx_{2n},Ty)}, \frac{d(Ax_{2n},Sx_{2n}).d(By,Ty)}{1+d(Sx_{2n},Ty)},\\ d(Sx_{2n},Ty)\end{Bmatrix}\right)$$

$$lim_{n\to\infty} d(Ax_{2n},By) \le k\,d(y,By)$$

$By = y = Ty$

Since $B(X) \subseteq S(X)$

for $w \in X$ such that $Sw = y$.

Now we show that $Aw = y$

$$d(Aw,By) \le k\max\left\{\begin{array}{c}\frac{d^2(Aw,Sw)+d^2(By,Ty)}{1+d(Sw,Ty)},\frac{d^2(Sw,By)+d^2(Aw,Ty)}{1+d(Sw,Ty)},\\ \frac{d(Aw,Sw)d(By,Ty)}{1+d(Sw,Ty)},\frac{d(Sw,By)d(Aw,Ty)}{1+d(Sw,Ty)}\\ d(Sw,Ty)\end{array}\right\} - \psi\left(\max\left\{\begin{array}{c}\frac{d^2(Aw,Sw)+d^2(By,Ty)}{1+d(Sw,Ty)},\frac{d^2(Sw,By)+d^2(Aw,Ty)}{1+d(Sw,Ty)},\\ \frac{d(Aw,Sw)d(By,Ty)}{1+d(Sw,Ty)},\frac{d(Sw,By)d(Aw,Ty)}{1+d(Sw,Ty)}\\ d(Sw,Ty)\end{array}\right\}\right)$$

It follows that, $d(Aw,y) \le kd(Aw,y)$

which contradiction, $d(Aw,y) > 0$ thus $Aw = y = Sw$
Since A and S are weakly compatible, so that $ASw = SAw$ this implies, $Ay = Sy$.
Now we show that, $Ay = y$ *for this* again from 3.1(iii),

$$d(Aw,By) \le k\max\left\{\begin{array}{c}\frac{d^2(Ay,Sy)+d^2(By,Ty)}{1+d(Sy,Ty)},\frac{d^2(Sy,By)+d^2(Ay,Ty)}{1+d(Sy,Ty)}\\ \frac{d(Ay,Sy)\,d(By,Ty)}{1+d(Sy,Ty)},\frac{d(Sy,By)d(Ay,Ty)}{1+d(Sy,Ty)}\\ d(Sy,Ty)\end{array}\right\} - \psi\left(\max\left\{\begin{array}{c}\frac{d^2(Ay,Sy)+d^2(By,Ty)}{1+d(Sy,Ty)},\frac{d^2(Sy,By)+d^2(Ay,Ty)}{1+d(Sy,Ty)}\\ \frac{d(Ay,Sy)\,d(By,Ty)}{1+d(Sy,Ty)},\frac{d(Sy,By)d(Ay,Ty)}{1+d(Sy,Ty)}\\ d(Sy,Ty)\end{array}\right\}\right)$$

it follows that, $d(Ay,y) \le k\,d(Ay,y)$

which contradiction thus $Ay = y$ and then, we write

$$Ay = Sy = By = Ty = y$$

that is y is common fixed point of $A, B, S, T.$ If $S(X)$ is closed subset of X then we follows similarly proof.

Uniqueness :- We suppose that x, is another fixed point for A, B, S, T then, by using 3.1(iii) then we have

$$d(x,y) \le k.d(x,y)$$

Which contradiction. *so that* $x = y$ and y is unique fixed point of $A, B, S, T.$
This complete the prove of the theorem.

Corollary 3.2. Let (X,d) be a $T-$ *orbitally* metric space, if A, B, S, T be the self mapping of X into itself such that;

3.2.(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, $T(X)$ or $S(X)$ are closed subset of X.
3.2.(ii) The pair (A,S) and (B,T) are weakly compatible and generalized weakly contractive map.
3.2.(iii) *for all* $x, y \in \overline{O(x_0)}$ *and* $k \in [0,1)$, *we define*,

$$d(Ax,By) \le k.M(x,y) - \psi(M(x,y))$$

Where , $M(Ax,By) = \max\left\{\begin{array}{c}\frac{d^2(Ax,Sx)+d^2(By,Ty)}{1+d(Sx,Ty)},\frac{d^2(Sx,By)+d^2(Ax,Ty)}{1+d(Sx,Ty)},\\ \frac{d(Ax,Sx).d(By,Ty)}{1+d(Sx,Ty)},\frac{d(Sx,By).d(Ax,Ty)}{1+d(Sx,Ty)},d(Sx,Ty)\end{array}\right\}.$

T
hen A, B, S, T have unique fixed point in $\overline{O(x_0)}$.

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