OPEN ACCESS

International
Journal
Of Modern Engineering Research (IJMER)

A Non Local Boundary Value Problem with Integral Boundary Condition

Goteti V R L Sarma[1], Awet Mebrahtu[2] Mebrahtom Sebhatu[3]
Department of Mathematics, Eritrea Institute of Technology, Mainefhi, Eritrea.

ABSTRACT: *In this article a three point boundary value problem associated with a second order differential equation with integral type boundary conditions is proposed. Then its solution is developed with the help of the Green's function associated with the homogeneous equation. Using this idea and Iteration method is proposed to solve the corresponding linear problem.*
Keywords: *Green's function, Schauder fixed point theorem, Vitali's convergence theorem.*
Ams Msc: *34B18, 34B99, 35J05*

I. INTRODUCTION

Non local boundary value problems raises much attention because of its ability to accommodate more boundary points than their corresponding order of differential equations [5], [8]. Considerable studies were made by Bai and Fag [2], Gupta [4] and Web [9]. This research article is concerned with the existence and uniqueness of solutions for the second order three-point boundary value problem with integral type boundary conditions

$-u'' = f(t, u(t)), \qquad t \in (a,b)$

$u'(a) = 0$

(1.1)

$$u'(b) + ku(\eta) = \int_a^b u(s)g(s)ds$$

where $f: [a,b] \times \mathbb{R} \to \mathbb{R}$ is a given function and $g: [a,b] \to \mathbb{R}$ is an integral function and $\eta \in (a,b)$. Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems.

The Green's function plays an important role in solving boundary value problems of differential equations. The exact expressions of the solutions for some linear ODEs boundary value problems can be denoted by Green's functions of the problems. The Green's function method might be used to obtain an initial estimate for shooting method. The Greens function method for solving the boundary value problem is an effect tools in numerical experiments. Some BVPs for nonlinear integral equations the kernels of which are the Green's functions of corresponding linear differential equations. The integral equations can be solved by to investigate the property of the Green's functions. The undetermined parametric method we use in this paper is a universal method, the Green's functions of many boundary value problems for ODEs can be obtained by similar method.

In (2008), Zhao discussed the solutions and Green's functions for non local linear second-order Three-point boundary value problems.

$u'' + f(t) = 0, \quad t \in [a,b]$

subject to one of the following boundary value conditions:

i. $u(a) = ku(\eta), \ \ u(b) = 0$ ii. $u(a) = 0, \quad u(b) = ku(\eta)$ iii. $u(a) = ku'(\eta), \quad u(b) = 0$

iv. $u(a) = 0, \ u(b) = ku'(\eta)$

where k was the given number and $\eta \in (a,b)$ is a given point.

In (2013), Mohamed investigate the positive solutions to a singular second order boundary value problem with more generalized boundary conditions. He consider the Sturm-Liouville boundary value problem

$u'' + \lambda g(t)f(t) = 0, \quad t \in [0,1]$ with the boundary conditions

$\propto u(0) - \beta u'(0) = 0, \ \gamma u(1) + \delta u'(1) = 0$

where $\propto > 0, \ \beta > 0, \ \gamma > 0 \ and \ \delta > 0$ are all constants, λ is a positive parameter and $f(.)$ Is singular at $u = 0$. Also the existence of positive solutions of singular boundary value problems of ordinary differential equations has been studied by many researchers such as **Agarwal and Stanek** established the existence criteria for positive solutions singular boundary value problems for nonlinear second order ordinary and delay differential

equations using the Vitali's convergence theorem. **Gatical et al** proved the existence of positive solution of the problem
$u^{''} + f(t) = 0, \quad t \in [0,1]$ with the boundary conditions
$\propto u(0) - \beta u^{'}(0) = 0\,, \;\; \gamma u(1) + \delta u^{'}(1) = 0$
using the iterative technique and fixed point theorem for cone for decreasing mappings.

Wang and Liu proved the existence of positive solution to the problem
$u^{''} + \lambda g(t) f(t) = 0, \quad t \in [0,1]$ with the boundary conditions $\propto u(0) - \beta u^{'}(0) = 0$ and $\gamma u(1) + \delta u^{'}(1) = 0$ using the Schauder fixed point theorem.

II. PRELIMINARIES

In this section, we introduce notations, definitions and preliminary facts that will be used in the remainder of this paper. Let $AC^1\big((a,b),\mathbb{R}\big)$ be the space of differentiable functions $u:(a,b) \to \mathbb{R}$ whose first derivative, u' is absolutely continuous.
We take $C([a,b],\mathbb{R})$ to be the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with norm $\|u\|_\infty = \sup\{|u(t)|: a \le t \le b\}$
And let $L^1([a,b],\mathbb{R})$ denote the Banach space of the functions $u:(a,b) \to \mathbb{R}$ that are Lesbegue integrable with norm $\|u\|_{L^1} = \int_a^b u(t)dt$

Definition 2.1: A map $f:[a,b] \times \mathbb{R} \to \mathbb{R}$ is said to be L^1- caratheodory if
(i) $t \to f(t,u)$ is measurable for each $u \in \mathbb{R}$
(ii) $u \to f(t,u)$ is continuous for almost each $t \in [a,b]$
(iii) For every $r > 0$ there exists $h_r \in L^1([a,b],\mathbb{R})$ such that $|f(t,u)| \le h_r(t)$ for almost each $t \in [a,b]$ and all $|u| \le r$

III. EXISTENCE AND UNIQUENESS RESULTS

Definition 3.1 A function $u \in AC^1\big((a,b),\mathbb{R}\big)$ is said to be a solution of (1.1) if u satisfies (1.1)
Lemma 3.1 : The Green's function of the corresponding homogeneous boundary value problem with homogeneous boundary conditions

$$-u'' = 0 \text{ satisfying } \quad u^{'}(a) = 0,\; u^{'}(b) + ku(\eta) = 0$$

is given by $G(t,s) = \begin{cases} \eta - s + \frac{1}{k}, & a \le t < s \\ \eta - t + \frac{1}{k}, & s \le t \le b \end{cases}$

Proof: It can be easily obtained from the elementary properties of Green's function hence omitted.
Assume that $g_* = \int_a^b \frac{1}{k} g(s)ds \ne 1$ and $k \ne 0$. One need the following auxiliary results.
Lemma 3.2: Let $f: L^1([a,b],\mathbb{R})$ then the function defined by $u(t) = \int_a^b H(t,s) f(s)ds$
is the unique solution of the boundary value problem

$$-u^{''} = f(t), \quad t \in (a,b) \text{ satisfying } u^{'}(a) = 0,\; u^{'}(b) + ku(\eta) = \int_a^b g(s)u(s)ds \tag{3.1}$$

where $H(t,s) = G(t,s) + \frac{1}{1-\int_a^b \frac{1}{k} g(s)ds}\left[\int_a^b u(r)f(r)dr + \int_a^b \int_a^b G(t,r)f(r)dr\right]ds$
and $G(t,s) = \begin{cases} \eta - s + \frac{1}{k}, & a \le t < s \\ \eta - t + \frac{1}{k}, & s \le t \le b \end{cases}$
Proof: Let u be a solution of the problem (3.1). Integrating we obtain

$$u^{'}(t) = u^{'}(a) - \int_a^t u(s)f(s)ds$$

$$u^{'}(a) = 0,\; \Rightarrow u^{'}(t) = -\int_a^t u(s)f(s)ds \tag{3.2}$$

$$u^{'}(b) = -\int_a^b u(s)f(s)ds$$

Integrating (3.2)

$$u(t) = u(a) - \int_a^t (t-s)f(s)ds$$

$$u(\eta) = u(a) - \int_a^\eta (\eta - s)f(s)ds$$

From the condition $u^{'}(b) + ku(\eta) = \int_a^b u(s)g(s)ds$

$$u(a) = \frac{1}{k}\int_a^b u(s)g(s)ds + \frac{1}{k}\int_a^b u(s)f(s)ds + \int_a^\eta (\eta - s)f(s)ds$$

Hence $u(t) = \frac{1}{k}\int_a^b u(s)g(s)ds + \frac{1}{k}\int_a^b u(s)f(s)ds + \int_a^\eta (\eta - s)f(s)ds + \int_a^t (s-t)f(s)ds$

$$u(t) = \frac{1}{k}\int_a^b u(s)g(s)ds + \frac{1}{k}\int_a^b u(s)f(s)ds + \int_a^b G(t,s)f(s)ds \qquad (3.3)$$

where $G(t,s) = \begin{cases} \eta - s + \frac{1}{k}, & a \le t < s \\ \eta - t + \frac{1}{k}, & s \le t \le b \end{cases}$

Multiply (3.3) by g(s) and integrating over (a,b)

$$\int_a^b u(s)g(s)ds = \int_a^b g(s)\left[\frac{1}{k}\int_a^b u(r)g(r)dr + \frac{1}{k}\int_a^b u(r)f(r)dr + \int_a^b G(s,r)f(r)dr\right]ds$$

$$\int_a^b u(s)g(s)ds\left[1 - \int_a^b \frac{1}{k}g(s)ds\right] = \int_a^b g(s)\left[\frac{1}{k}\int_a^b u(r)f(r)dr + \int_a^b G(s,r)f(r)dr\right]ds$$

$$\int_a^b u(s)g(s)ds = \frac{\int_a^b g(s)\left[\frac{1}{k}\int_a^b u(r)f(r)dr + \int_a^b G(s,r)f(r)dr\right]ds}{1 - \int_a^b \frac{1}{k}g(s)ds} \qquad (3.4)$$

Substituting (3.4) in (3.3) gives

$$u(t) = \int_a^b G(t,s)f(s)ds + \frac{1}{k}\int_a^b u(s)f(s)ds + \frac{1}{k}\frac{\int_a^b g(s)\left[\frac{1}{k}\int_a^b u(r)f(r)dr + \int_a^b G(s,r)f(r)dr\right]ds}{1 - \int_a^b \frac{1}{k}g(s)ds}$$

Therefore $u(t) = \int_a^b H(t,s)f(s)ds$. Hence it is proved

Now let us set $g^* = |1 - g_*|$. Note that

$$|G(t,s)| \le \frac{1}{k} + \frac{a+b}{2},$$

And $if\ k \ge 1 \quad |G(t,s)| \le 1 + \frac{a+b}{2} \quad for\ (t,s) \in [a,b] \times [a,b]$

Theorem 3.3: Assume that f is an $L^1 - Caratheodory$ function and the following hypothesis

(A1) there exists $l \in L^1([a,b],\mathbb{R}^+)$ such that

$|f(t,x) - f(t,\bar{x})| \le l(t)|x - \bar{x}|, \quad \forall\, x,\bar{x} \in \mathbb{R}, \quad t \in [a,b]$

Holds. If $\|l\|_{L^1} + \frac{1}{k}\|g\|_{L^1} + \frac{\|g\|_{L^1}}{kg^*}\left[\frac{1}{k}\|g\|_{L^1} + \|l\|_{L^1}\right] < 1 + \frac{2}{a+b}$

Then the BVP (1.1) has a unique solution

Proof :

Transform problem (1.1) into a fixed point problem consider the operator

$N: C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ defined by

$N(u(t)) = \int_a^b H(t,s)f(s,u(s))ds \quad t \in [a,b]$

We will show that N is a contraction. Indeed, consider $u,\bar{u} \in C([a,b],\mathbb{R})$ then we have each $t \in [a,b]$.

$$|N(u(t)) - N(\bar{u}(t))| \le \int_a^b |H(t,s)||f(s,u(s)) - f(s,\bar{u}(s))|ds$$

$$\le \int_a^b |G(t,s)|l(s)|u(s) - \bar{u}(s)|ds + \frac{1}{k}\int_a^b l(s)|u(s) - \bar{u}(s)|^2 ds + \frac{1}{kg^*}\left[\int_a^b \frac{1}{k} l(s)|u(s) -\right.$$

us2grdsdr+ablsus−usgrabGr,sdsdr

Boundary value problems

Therefore

$$\|N(u) - N(\bar{u})\|_\infty \le \left(1 + \frac{a+b}{2}\right)\left(\|l\|_{L^1} + \|g\|_{L^1}{}^2 + \frac{\|g\|_{L^1}{}^2 + \|g\|_{L^1}\cdot\|l\|_{L^1}}{g^*}\right)\|u - \bar{u}\|$$

Showing that, N is a contraction and hence it has a unique fixed point which is a solution to (1.1). The proof is completed

We now present an existence result for problem (1.1)

Theorem 3.4: Suppose that the hypothesis
(H1) The function $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is an $L^1-caratheodory$
(H2) there exist functions $p,\bar{q}\in L^1([a,b],\mathbb{R}+)$ and $\propto\in[a,b]$ such that
$|f(t,u)|\le p(t)|u|^{\propto}-\bar{q}(t),\ \forall (t,u)\in[a,b]\times\mathbb{R}$
are satisfied. Then the BVP (1.1) has at least one solution.
Moreover the solution set $s=\{\{u\in C([a,b],\mathbb{R}): u\ solution\ of\ the\ problem\ (1.1)\}\}$ is compact.

Proof: Transform the BVP (1.1) into a fixed-point problem. Consider the operator N as define in theorem 3.3
We will show that N satisfies the assumptions of the nonlinear alternative of Leray-schauder type.
The proof will be given in several steps
Step1. (N is continuous). Let $\{u_m\}$ be a sequence such that $u_m\to u\in C([a,b],\mathbb{R})$. Then
$\|N(u_m)-N(u)\|_\infty\le\int_a^b|H(t,s)|\left|f(s,u_m(s))-f(s,u(s))\right|ds$
Since f is $L^1-caratheodory$ and $g\in L^1([a,b],\mathbb{R}^+)$, then

$$\|N(u_m)-N(u)\|_\infty \le\left(1+\frac{a+b}{2}\right)\left\|f(.,u_m(.))-f(.,u(.))\right\|_{L^1}+\left(1+\frac{a+b}{2}\right)\|g\|_{L^1} +\left(\frac{a+b+2}{2g^*}\right)\left(\|g\|_{L^1}+\left\|f(.,u_m(.))-f(.,u(.))\right\|_{L^1}\right)$$

Hence $\|N(u_m)-N(u)\|_\infty\to 0\ as\ m\to\infty$
Step 2 (N maps bounded sets into bounded sets in $C([a,b],\mathbb{R})$). Indeed, it is enough to show that there exists a positive constant l such that for each $u\in B_q=\{u\in C([a,b],\mathbb{R});\|u\|_\infty\le q\}$. one has $\|u\|_\infty\le l$
Let $u\in B_q$. Then for each $t\in[a,b]$, we have $N(u(t))=\int_a^b H(t,s)f(s,u(s))ds$
By (H2) we have for each $t\in[a,b]$

$$N(u(t))\le\int_a^b H(t,s)f(s,u(s))ds$$
$$\le\left(1+\tfrac{a+b}{2}\right)[\|\bar{q}\|_{L^1}+q^{\propto}\|p\|_{L^1}]+\left(1+\tfrac{a+b}{2}\right)\|\bar{q}\|_{L^1}+\left(1+\tfrac{a+b}{2}\right)\tfrac{\|g\|_{L1}}{g^*}[\|\bar{q}\|_{L^1}+q^{\propto}\|p\|_{L^1}]=l$$

Then for each $u\in B_q$ we have
$\|N(u)\|_\infty\le l$
Step 3 (N maps bounded set into equicontinuous sets of $C([a,b],\mathbb{R})$). Let $\tau_1,\tau_2\in[a,b]$,
$\tau_1<\tau_2$ and B_q be a bounded set of $C([a,b],\mathbb{R})$ as in step 2. Let $u\in B_q$ and $\in[a,b]$ we have
$\left|N(u(\tau_2))-N(u(\tau_1))\right|\le\int_a^b|H(\tau_2,s)-H(\tau_1,s)|\bar{q}(s)ds+q^{\propto}\int_a^b|H(\tau_2,s)-H(\tau_1,s)|p(s)ds$
As $\tau_2\to\tau_1$ the right hand side of the above inequality tends to zero. Then $N(B_q)$ is equicontinuous. As a consequence of step 1 to 3 together with the Arzela-Ascoli theorem we can conclude that $N: C([a,b],\mathbb{R})\to C([a,b],\mathbb{R})$ is completely continuous.
Step 4. (A priori bounds on solutions). Let $u=\gamma N(u)$ for some $a<\gamma<b$. this implies by (H2) that each $t\in[a,b]$ we have
$u(t)\le\left(1+\frac{a+b}{2}\right)(\int_a^b p(s)|u(s)|^{\propto}ds+\|\bar{q}\|_{L^1}+\frac{\|g\|_{L1}}{g^*}\|\bar{q}\|_{L^1}+\frac{\|g\|_{L1}}{g^*}\int_a^b p(s)|u(s)|^{\propto}ds)$

Then $\|u\|_\infty\le\left(1+\frac{a+b}{2}\right)\left(\|p\|_{L^1}\|u\|_\infty^{\propto}+\|\bar{q}\|_{L^1}+\frac{\|g\|_{L1}}{g^*}\|\bar{q}\|_{L^1}+\frac{\|g\|_{L1}}{g^*}\|p\|_{L^1}\|u\|_\infty^{\propto}\right)$
Boundary value problems
If $\|u\|_\infty>b$ we have
$\|u\|_\infty^{b-\propto}\le\left(1+\frac{a+b}{2}\right)\left(\|p\|_{L^1}+\|\bar{q}\|_{L^1}+\frac{\|g\|_{L1}}{g^*}\|\bar{q}\|_{L^1}+\frac{\|g\|_{L1}}{g^*}\|p\|_{L^1}\right)$
Thus $\|u\|_\infty\le((1+\frac{a+b}{2})\left(\|p\|_{L^1}+\|\bar{q}\|_{L^1}+\frac{\|g\|_{L1}}{g^*}\|\bar{q}\|_{L^1}+\frac{\|g\|_{L1}}{g^*}\|p\|_{L^1}\right))^{\frac{1}{b-\propto}}=\varphi_*$
Hence $\|u\|_\infty\le\max(b,\varphi)=M$
Set $y=\{u\in C([a,b],\mathbb{R}):\|u\|_\infty<M+b\}$
And consider the operator $N:\bar{y}\to\in C([a,b],\mathbb{R})$. from the choice of y, there is no $u\in\partial y$ such that $u=\gamma N(u)$ for some $\gamma\in(a,b)$. we deduce that N has a fixed point u in y which is a solution of the problem (1.1)
$u_m=\int_a^b H(t,s)f(s,u_m(s))ds,\quad m\ge1,\quad t\in(a,b)$
As in step 3 and 4 we can easily prove that there exists $M>a$ such that
$\|u_m\|_\infty<M,\quad\forall m\ge1$

And the set $\{u_m, m \geq 1\}$ is equicontinuous in $C([a,b],\mathbb{R})$ hence by Arzela-Ascoli theorem we can conclude that there exists a subsequence of $\{u_m, m \geq 1\}$ converging to u in $C([a,b],\mathbb{R})$ using that fact that f in an $L^1 - Caratheodory$ we can prove that

$u(t) = \int_a^b H(t,s) f(s,u(s)) ds, \quad t \in (a,b)$ thus s is compact.

REFERENCES

[1]. Al-Hayan, W. (2007). A domain Decomposition method with Green's functions for solving Twelfth-order of boundary value problems. Applied Mathematical sciences, Vol. 9, 2015, no. 8, 353 – 368.

[2]. Bender, C. M. and S. A. Orzag (1999). Advanced mathematical methods for scientists and Engineers; Asymptotic methods and perturbation theory, ACM30020.

[3]. Benchohra, M. at al Second-Order boundary value problem with integral boundary conditions. Boundary value problems article ID 260309 vol. 2011

[4]. Dr. Raisinghania, M. D. (2013), Integral equations and boundary value problems sixth edition; S. Chand & Company PVT. LTD.

[5]. Greengard, L. and V. Kokhlin (1991). On the numerical solution of two-point boundary value problems. Communications on pure and applied mathematics vol. XLIV, 419-452(1991)

[6]. Herron, I. H. Solving singular boundary value problems for ordinary differential equations. Caribb. J. Math. Comput. Sci. 15, 2013, 1- 30.

[7]. Kumlin, P. (2003/2004), A note on ordinary differential equations; TMA401/MAN 670 Functional Analysis. Mathematics Chalmers & GU

[8]. Liu, Z., Kang, S. M and J. S. Ume (2009). Triple positive solutions of nonlinear third order boundary value problems. Taiwanese Journal of Mathematics Vol. 13, no. 3 pp955-971.

[9]. Mohamed, M. & W. A. W. Azmi (2013), positive solutions to solutions to a singular second order boundary value problems. Int. Journal of math. Analysis, Vol.7, 2013, no. 41, 2005-2017.

[10]. Raisinghania, M. D. (2011), Integral equations and boundary value problems sixth edition; S. Chand & Company PVT. LTD. New Delthi-110 055.

[11]. Teterina, A. O. (2013), The Green's function method for solutions of fourth order nonlinear boundary value problem. The university of Tennessee, Knoxville

[12]. Yang, C. & P. Weng (2007). Green's function and positive solutions for boundary value problems of third order differential equations. Computers and mathematics with applications 54(2007)567-578.

[13]. Zhao, Z. (2007) positive solutions for singular three-point boundary value problems. Electronic Journal of Differential equations Vol. 2007(2007), no. 156, pp. 1-8. ISSN 1072-6691

[14]. Zhao, Z. (2007), Solution and green's functions for linear second order three-point boundary value problems; Computers and mathematics with applications 56(2008)104-113.