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Some Identities Relating Mock Theta Functions And Hikami Mock Theta Functions

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Abstract: *In this paper, we established certain new representations of Mock theta functions and Hikami's Mock theta functions by making use of identities due to Singh, S. P. and Mishra, B. P. [12].*

Keywords: *Mock theta functions of order two, six, eight and ten, Hikami's Mock theta functions of order two, four and eight, identities.*

I. INTRODUCTION

Srinivasa Ramanujan submitted an interesting function defined by a $q-$series as a gift to the world of mathematics is known as Mock theta function. He gave a list of seventeen such functions classifying them as of order three, five and seven, but did not say what he meant by order. Furthermore after the discovery of Ramanujan's Lost Notebook more Mock theta functions of distinct orders studied by Andrews, G. E. [2], Andrews, G. E. and Hicerson, D. [3], Gordon, B. and McIntosh, R., J. [4,5] and Choi, Y. S. [6]. Hikami's [8, 9] discovered new Mock theta functions of order two, order four and order eight respectively. Recently, Eltikali, M., Paul, A. et al. [7] developed new representation among Hikami Mock theta functions related with many selected identities. If $F(q) := \sum_{n=0}^{\infty} f(q,n)$ is a Mock theta function, then the corresponding partial Mock theta function is denoted by the truncated series, $F_m(q) := \sum_{n=0}^{m} f(q,n)$ and the complete Mock theta function is defined by the bilateral series $F_m(q) := \sum_{n=-\infty}^{\infty} f(q,n)$.

II. DEFINITIONS AND NOTATIONS

Throughout this article, we assume that $|q|<1$ and use the notation

$$(a)_0 := (a;q) := 1, \qquad (a;q^k)_0 := 1 \tag{2.1}$$

$$(a)_n := (a;q)_n := \prod_{r=0}^{n-1}(1-aq^r), \qquad n \geq 1. \tag{2.2}$$

$$(a;q)_\infty := \prod_{r=0}^{\infty}(1-aq^r), \qquad (a;q^k)_n := \prod_{r=0}^{n-1}(1-aq^{kr}), \tag{2.3}$$

where $(a;q^k)_n$ is $q-$shifted factorial.

The second order Mock theta functions are:

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}, \qquad B(q) := \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}$$

(2.4)

$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2}.$$

The last function $\mu(q)$ appears in Ramanujan's "Lost" Notebook.

The six order Mock theta functions are:

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}, \qquad \psi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}};$$

,

$$\rho(q) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}(-q;q)_n}{(q;q^2)_{n+1}}, \qquad \sigma(q) := \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)(n+2)}{2}}(-q;q)_n}{(q;q^2)_{n+1}}$$

(2.5)

$$\lambda(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^n(q;q^2)_n}{(-q;q)_n}, \qquad \mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n(q;q^2)_n}{(-q;q)_n},$$

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n}.$$

The eight order Mock theta functions are:

$$S_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n} \qquad S_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n}$$

$$T_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n} \qquad T_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n}$$

$$U_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n} \qquad U_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n}$$

$$V_0(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} \qquad V_1(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} \qquad (2.6)$$

The ten order Mock theta functions are:

$$\phi_{LC}(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}, \qquad \psi_{LC}(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}}$$

$$X_{LC}(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}, \qquad \chi_{LC}(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}. \tag{2.7}$$

Hikami's Mock theta functions are:

$$D_5(q) := \sum_{n=0}^{\infty} \frac{q^n(-q;q)_n}{(q;q^2)_{n+1}}, \qquad D_6(q) := \sum_{n=0}^{\infty} \frac{q^n(-q^2;q^2)_n}{(q^{n+1};q)_{n+1}},$$

$$I_{12}(q) := \sum_{n=0}^{\infty} \frac{q^{2n}(-q;q^2)_n}{(q^{n+1};q)_{n+1}}, \qquad I_{13}(q) := \sum_{n=0}^{\infty} \frac{q^n(-q;q^2)_n}{(q^{n+1};q)_{n+1}}. \tag{2.8}$$

We shall make use of following identities in our analysis:

$$\frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty} \sum_{n=0}^{\infty} \alpha_n := \sum_{n=0}^{\infty} \frac{[a,y;q]_n q^n}{[q,ayq;q]_n} \sum_{r=0}^{n} \alpha_r + \sum_{n=0}^{\infty} \frac{[aq,yq;q]_n}{[q,ayq;q]_n} \alpha_{n+1}. \tag{2.9}$$

$$\sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r := \sum_{n=0}^{\infty} \delta_n \sum_{r=0}^{n} \alpha_r + \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r - \sum_{n=0}^{\infty} \alpha_n \delta_n. \tag{2.10}$$

$$\sum_{n=-\infty}^{\infty} \alpha_n \sum_{r=0}^{\infty} \delta_r := \sum_{n=0}^{\infty} \delta_n \sum_{r=-n}^{n} \alpha_r + \sum_{n=-\infty}^{\infty} \alpha_n \sum_{r=0}^{n} \delta_r - \sum_{n=-\infty}^{\infty} \alpha_n \delta_n. \tag{2.11}$$

III. MAIN RESULTS

1. We establish new representations involving partial Mock theta functions of order two, six, eight and ten. Also, partial Hikami's Mock theta functions of order two, four and eight are obtained.

$$\frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty} A(q) = \sum_{n=0}^{\infty} \frac{[a,y;q]_n q^n}{[q,ayq;q]_n} A_n(q) + \sum_{n=0}^{\infty} \frac{[aq,yq;q]_n}{[q,ayq;q]_n} \frac{q^{n+2}(-q^2;q^2)_{n+1}}{(q;q^2)_{n+2}}. \tag{3.1}$$

Where $A(q)$ is a Mock theta function of order two.

$$\frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty} \rho(q) = \sum_{n=0}^{\infty} \frac{[a,y;q]_n q^n}{[q,ayq;q]_n} \rho_n(q) + \sum_{n=0}^{\infty} \frac{[aq,yq;q]_n}{[q,ayq;q]_n} \frac{q^{(n+1)(n+2)/2}(-q;q)_{n+1}}{(q;q^2)_{n+2}}. \tag{3.2}$$

Where $\rho(q)$ is a Mock theta function of order six.

$$\frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty} S_0(q) = \sum_{n=0}^{\infty} \frac{[a,y;q]_n q^n}{[q,ayq;q]_n} S_{0,N}(q) + \sum_{n=0}^{\infty} \frac{[aq,yq;q]_n}{[q,ayq;q]_n} \frac{q^{(n+1)^2}(-q;q^2)_{n+1}}{(-q^2;q^2)_{n+1}}. \tag{3.3}$$

Where $S_0(q)$ is a Mock theta function of order eight.

$$\frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty}\phi_{LC}(q)=\sum_{n=0}^{\infty}\frac{[a,y;q]_n q^n}{[q,ayq;q]_n}\phi_{LC,n}(q)+\sum_{n=0}^{\infty}\frac{[aq,yq;q]_n}{[q,ayq;q]_n}\frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+2}}. \tag{3.4}$$

Where $\phi_{LC}(q)$ is a Mock theta function of order ten.

$$\frac{[aq,yq;q]_\infty}{[q,ayq;q]_\infty}D_5(q)=\sum_{n=0}^{\infty}\frac{[a,y;q]_n q^n}{[q,ayq;q]_n}D_{5,n}(q)+\sum_{n=0}^{\infty}\frac{[aq,yq;q]_n}{[q,ayq;q]_n}\frac{q^{(n+1)}(-q;q)_{n+1}}{(q;q^2)_{n+2}}. \tag{3.5}$$

Where $D_5(q)$ is Hikami's Mock theta function of order two.

2.In this section we shall make use of identity (2.10) to establish product formulae for any two Mock theta functions.

$$A(q)B(q)=\sum_{n=0}^{\infty}\frac{q^{n+1}(-q;q^2)_n}{(q;q^2)_{n+1}}A_n(q)+\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}B_n(q)-\sum_{n=0}^{\infty}\frac{q^{2(n+1)}(-q^2;q^2)_n(-q;q^2)_n}{(q;q^2)_{n+1}^2} \tag{3.6}$$

This is a product formula for two Mock theta functions of order two.

$$A(q)\rho(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}}A_n(q)+\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}\rho_n(q)-\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)/2}(-q;q)_n(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} \tag{3.7}$$

This is a product formula for Mock theta functions of order two and six.

$$A(q)S_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_{n+1}}A_n(q)+\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}S_{0,n}(q)-\sum_{n=0}^{\infty}\frac{q^{n(n+1)+1}(-q;q^2)_n}{(q;q^2)_{n+1}}. \tag{3.8}$$

This is a product formula for Mock theta functions of order two and eight.

$$A(q)\phi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}A_n(q)+\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}\phi_{LC,n}(q)-\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)/2}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} \tag{3.9}$$

This is a product formula for Mock theta functions of order two and ten.

$$A(q)D_5(q)=\sum_{n=0}^{\infty}\frac{q^n(-q;q)_n}{(q;q^2)_{n+1}}A_n(q)+\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}D_{5,n}(q)-\sum_{n=0}^{\infty}\frac{q^{2n+1}(-q;q)_n(-q^2;q^2)_n}{(q;q^2)_{n+1}^2}. \tag{3.10}$$

This is a product formula for Mock theta functions of order two and Hikami's Mock theta function of ordero.

$$\phi(q)\psi(q)=\sum_{n=0}^{\infty}\frac{(-)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}}\phi_n(q)+\sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}\psi_n(q)-\sum_{n=0}^{\infty}\frac{(-)^{2n}q^{2n(n+1)+1}(q;q^2)_n^2}{(-q;q)_{2n}(-q;q)_{2n+1}}. \tag{3.11}$$

This is a product formula for two Mock theta functions of order six.

$$\phi(q)S_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}\phi_n(q)+\sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}S_{0,n}(q)-\sum_{n=0}^{\infty}\frac{(-)^n q^{2n^2}(q;q^2)_n}{(-q^2;q^2)_n}. \tag{3.12}$$

This is a product formula for Mock theta functions of order six and eight.

$$\phi(q)\phi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}\phi_n(q)+\sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}\phi_{LC,n}(q)-\sum_{n=0}^{\infty}\frac{(-)^n q^{n(3n+1)/2}(q;q^2)_n}{(q;q^2)_{n+1}(-q;q)_{2n}}. \tag{3.13}$$

This is a product formula for Mock theta functions of order six and ten.

$$\phi(q)D_5(q)=\sum_{n=0}^{\infty}\frac{q^n(-q;q)_n}{(q;q^2)_{n+1}}\phi_n(q)+\sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}}D_{5,n}(q)-\sum_{n=0}^{\infty}\frac{(-)^n q^{n(n+1)}(q;q^2)_n(-q;q)_n}{(-q;q)_{2n}(q;q^2)_{n+1}}. \tag{3.14}$$

This is a product formula for Mock theta functions of order six and Hikami's Mock theta function of order two.

$$S_0(q)S_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n}S_{0,n}(q)+\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}S_{1,n}(q)-\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q;q^2)_n^2}{(-q^2;q^2)_n^2}. \tag{3.15}$$

This is a product formula for two Mock theta functions of order eight.

$$S_0(q)\phi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-q;q^2)_n}{(q;q^2)_{n+1}}S_{0,n}(q)+\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}\phi_{LC,n}(q)-\sum_{n=0}^{\infty}\frac{q^{n(3n+1)/2}(-q;q^2)_n}{(q;q^2)_{n+1}(-q^2;q^2)_n}. \tag{3.16}$$

This is a product formula for Mock theta functions of order eight and ten.

$$S_0(q)D_5(q)=\sum_{n=0}^{\infty}\frac{q^n(-q;q)_n}{(q;q^2)_{n+1}}S_{0,n}(q)+\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}D_{5,n}(q)-\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q;q^2)_n(-q;q)_n}{(-q^2;q^2)_n(q;q^2)_{n+1}}. \tag{3.17}$$

This is a product formula for Mock theta functions of order eight and Hikami's Mock theta function of order two.

$$\phi_{LC}(q)\psi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)/2}(-q;q^2)_n}{(q;q^2)_{n+1}}\phi_{LC,n}(q)+\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}\psi_{LC,n}(q)-\sum_{n=0}^{\infty}\frac{q^{n(n+2)+1}}{(q;q^2)_{n+1}^2}. \tag{3.18}$$

This is a product formula for two Mock theta functions of order ten.

$$\phi_{LC}(q)D_5(q)=\sum_{n=0}^{\infty}\frac{q^n(-q;q)_n}{(q;q^2)_{n+1}}\phi_{LC,n}(q)+\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}D_{5,n}(q)-\sum_{n=0}^{\infty}\frac{q^{n(n+3)/2}(-q;q)_n}{(q;q^2)_{n+1}^2}. \tag{3.19}$$

This is a product formula for Mock theta functions of order ten and Hikami's Mock theta function of order two.

$$D_5(q)D_6(q)=\sum_{n=0}^{\infty}\frac{q^n(-q^2;q^2)_n}{(q^{n+1};q)_{n+1}}D_{5,n}(q)+\sum_{n=0}^{\infty}\frac{q^n(-q;q)_n}{(q;q^2)_{n+1}}D_{6,n}(q)-\sum_{n=0}^{\infty}\frac{q^{2n}(-q;q)_n(-q^2;q^2)}{(q;q^2)_{n+1}(q^{n+1};q)_{n+1}}. \tag{3.20}$$

This is a product formula for two Hikami's Mock theta function of order two and order four.

3.Finally, we establish product formulae among complete Mock theta function of order two and Mock theta function of distinct orders by making use of identity (2.11).

$$A_C(q)B(q)=\sum_{n=0}^{\infty}\frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}A_{(-n,n)}(q)+\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}B_n(q)-\sum_{n=-\infty}^{\infty}\frac{q^{2n+1}}{(q;q^2)_{n+1}^2}. \qquad (3.21)$$

where $A_C(q)$ is complete Mock theta function of order two and $B(q)$ is another Mock theta function of order two.

$$A_C(q)\phi(q)=\sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}(q;q^2)_n}{(-q;q^2)_{2n}}A_{(-n,n)}(q)+\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}\phi_n(q)-\sum_{n=-\infty}^{\infty}\frac{(-)^n q^{n(n+1)}(q;q^2)_n(-q^2;q^2)_n}{(-q;q)_{2n}(q;q^2)_{n+1}}$$

(3.22)

Where $\phi(q)$ is Mock theta function of order six.

$$A_C(q)S_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n}A_{(-n,n)}(q)+\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}S_{0,n}(q)-\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)+1}(-q;q^2)_n}{(q;q^2)_{n+1}}.$$

(3.23)

Where $S_0(q)$ is Mock theta function of order eight.

$$A_C(q)\phi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}A_{(-n,n)}(q)+\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}\phi_{LC,n}(q)-\sum_{n=-\infty}^{\infty}\frac{q^{(n+1)(n+2)/2}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2}$$

(3.24)

Where $\phi_{LC}(q)$ is Mock theta function of order ten.

$$A_C(q)D_5(q)=\sum_{n=0}^{\infty}\frac{q^n(-q;q)_n}{(q;q^2)_{n+1}}A_{(-n,n)}(q)+\sum_{n=0}^{\infty}\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}D_{5,n}(q)-\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)}(-q;q)_n(-q^2;q^2)_n}{(q;q^2)_{n+1}^2}.$$

(3.25)

Where $D_5(q)$ is Hikami's Mock theta function of order two.

IV. PROOF OF MAIN RESULTS

1.0 As an illustraction, we shall prove the results (3.1) to (3.5). In order to prove the result (3.1), taking $\alpha_n=\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}$ in the identity (2.9) and then after simplification, we obtain the result (3.1). Similarly, suitable selections of α_n and making use of respective identity (2.9) one can establish the results (3.2)-(3.5). Other Mock theta functions of order two, six eight, ten and Hikami's Mock theta functions can be expressed similarly by proper choice of α_n in identity (2.9).

2.As an illustration, we shall prove the result (3.6). Taking $\alpha_n=\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}$ and $\delta_n=\frac{q^{n+1}(-q;q^2)_n}{(q;q^2)_{n+1}}$ in identity (2.10) and then after simplification, we get the result (3.6). Choosing properly the values of α_n and δ_n and proceeding as above, one can obtain the results (3.7) to (3.20).

3.finally by making use of the identity (2.11) in order to prove the product formulae for Mock theta functions. As an illustration, we prove the result (3.21). Taking $\alpha_n=\frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}$ and $\delta_n=\frac{q^n(-q;q^2)_n}{(q;q^2)_{n+1}}$ in identity (2.11) and then after simplification, we get the result (3.6).\\By choosing the value of α_n and δ_n and proceeding as above, one can obtain the results (3.22) to (3.25).

V. APPLICATIONS

Choosing $a = y = -1$ in (3.1), we get

$$\frac{[-q;q]_\infty^2}{[q;q]_\infty^2} A(q) = \sum_{n=0}^{\infty} \frac{[-1;q]_n^2 q^n}{[q;q]_n^2} A_n(q) + \sum_{n=0}^{\infty} \frac{[-q;q]_n^2}{[q;q]_n^2} \frac{q^{n+2}(-q^2;q^2)_{n+1}}{(q;q^2)_{n+2}}. \tag{5.1}$$

Choosing $a = y = -1$ in (3.2), we get

$$\frac{[-q;q]_\infty^2}{[q;q]_\infty^2} \rho(q) = \sum_{n=0}^{\infty} \frac{[-1;q]_n^2 q^n}{[q;q]_n^2} \rho_n(q) + \sum_{n=0}^{\infty} \frac{[-q;q]_n^2}{[q;q]_n^2} \frac{q^{(n+1)(n+2)/2}(-q;q)_{n+1}}{(q;q^2)_{n+1}}. \tag{5.2}$$

Choosing $a = y = -1$ in (3.3), we get

$$\frac{[-q;q]_\infty^2}{[q;q]_\infty^2} S_0(q) = \sum_{n=0}^{\infty} \frac{[-1;q]_n^2 q^n}{[q;q]_n^2} S_{0,n}(q) + \sum_{n=0}^{\infty} \frac{[-q;q]_n^2}{[q;q]_n^2} \frac{q^{(n+1)^2}(-q;q^2)_{n+1}}{(-q^2;q^2)_{n+1}}. \tag{5.3}$$

Choosing $a = y = -1$ in (3.4), we get

$$\frac{[-q;q]_\infty^2}{[q;q]_\infty^2} \phi_{LC}(q) = \sum_{n=0}^{\infty} \frac{[-1;q]_n^2 q^n}{[q;q]_n^2} \phi_{LC,n}(q) + \sum_{n=0}^{\infty} \frac{[-q;q]_n^2}{[q;q]_n^2} \frac{q^{(n+1)(n+2)/2}(-q;q^2)_{n+1}}{(q;q^2)_{n+2}}. \tag{5.4}$$

Choosing $a = y = -1$ in (3.5), we get

$$\frac{[-q;q]_\infty^2}{[q;q]_\infty^2} D_5(q) = \sum_{n=0}^{\infty} \frac{[-1;q]_n^2 q^n}{[q;q]_n^2} D_{5,n}(q) + \sum_{n=0}^{\infty} (1+q^{n+1}) \frac{q^{n+1}(-q;q)_n}{(q;q^2)_{n+2}}. \tag{5.5}$$

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