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Fixed Point Theorems For Multiplicative $\psi - \beta$ Geraghty Contractions.

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ABSTRACT: In this paper we introduce the notion of multiplicative $\psi - \beta$ Geraghty Contraction on a multiplicative metric space and prove a fixed point theorem for such contractions.

KEY WORDS : Rational inequalities, Multiplicative metric space, multiplicative contraction, Geraghty contraction, multiplicative $\psi - \beta$ Geraghty Contraction, Common fixed point.

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I. INTRODUCTION AND PRELIMINARIES

In 1922, Banach [3] proved a theorem which is well known as "Banach's Fixed point theorem" to establish the existence and uniqueness of fixed point of a contractive mapping in a complete metric space. This principle is applicable to variety of subjects such as integral equations, differential equations, image processing and many others. The study on the existence of fixed points of some mappings satisfying certain contractions has many applications and has been the center of various research activities. In the past years, many authors generalized Banach's fixed point theorems in various spaces such as Quasi-metric spaces, Fuzzy metric spaces, Partial metrics paces and generalized metric spaces [2,6,14,15].

On the other hand, in 2008, Bashirov.A.E et al.[4] defined a new distance so called a multiplicative distance by using the concepts of multiplicative absolute value. In 2012, Florack.L and Assen.H.V [7] had shown the use of the concept of multiplicative calculus in biomedical image analysis. In 2012, Ozavsar.M and Cevikel.A.C [17] investigated multiplicative metric spaces by remarking its topological properties, and introduced concept of multiplicative contraction mapping and proved some fixed point theorems of multiplicative contraction mappings of multiplicative metric space.

Recently He.X, Song.M and Chen.D [9] proved common fixed point theorems for four self mappings in a multiplicative metric space. In 2016, Nisha Sharma, Kamal Kumar and Sharma.S [15] proved the common fixed point theorem between the self mappings using a rational contractive condition.

Motivated by the above result, in this paper, we improve the result of [15], and prove common fixed point theorem satisfying Geraghty type condition in a multiplicative metric space with rational inequalities. The letter R^+ denote the set of all positive real numbers.

Definition 1.1. (Bashirov. A. E., Kurpinar. E. M., Ozyapici. A [4]). Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathrm{R}^+$ satisfying the following conditions:

(i) $d(x,y) \geq 1$ for all $x, y \in X$ and $d(x,y) = 1$, if and only if $x = y$.

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$.

(iii) $d(x,y) \leq d(x,z).d(z,y)$ for all $x, y, z \in X$. (Multiplicative triangle inequality)

Also (X,d) is called a multiplicative metric space.

Note that R^+ is a multiplicative metric space with respect to the multiplication.

Example 1.2. (Ozavser. M., Cevikel. A. C. [17]). Let $d^*:(R^+)^n \times (R^+)^n \to R^+$ be defined as follows

$$d^*(x,y) = \left|\frac{x_1}{y_1}\right|^* . \left|\frac{x_2}{y_2}\right|^* ...... \left|\frac{x_n}{y_n}\right|^* .$$

where $x=(x_1,x_2,...x_n), y=(y_1,y_2,...y_n) \in R^+$ and $|.|^* : R^+ \to R^+$ is

$$|a|^* = \begin{cases} a & \text{if } a \geq 1 \\ \frac{1}{a} & \text{if } a \leq 1 \end{cases}$$ Then $((R^+)^n, d^*)$ is a multiplicative metric space.

Example 1.3. (Ozavser. M., Cevikel. A. C. [17]). Let $a>1$ be fixed real number. Then $d_a : R^n \to R^n$ is defined by $d_a(w,z) = a^{\sum_{i=1}^{n}|w_i - z_i|}$ where $w=(w_1,w_2,...w_n), z=(z_1,z_2,...z_n) \in \mathbb{R}^n$.

Obviously, $(\mathbb{R}^n, d_a)$ is a multiplicative metric space. We can also extended multiplicative metric C^n by the following definition: $d_a(w,z) = a^{\sum_{i=1}^{n}|w_i - z_i|}$ where $w=(w_1,w_2,...w_n), z=(z_1,z_2,...z_n) \in C^n$.

Definition 1.4. (Ozavser. M., Cevikel. A. C. [17]). (Multiplicative convergence). Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x) = \{y / d(x,y) < \varepsilon\}, \varepsilon > 1$ there exists a natural number N such that for $n \geq N$, $x_n \in B_\varepsilon(x)$, the sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n \to x \ (n \to \infty)$.

Definition 1.5. (Ozavser. M., Cevikel. A. C. [17]). Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if, for each $\varepsilon > 1$, there exists $N \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon$, for all $m,n \geq N$.

Definition 1.6. (Ozavser. M., Cevikel. A. C. [17]). Let (X,d) be a multiplicative metric space. We call (X,d) is complete if every multiplicative Cauchy sequence in X is multiplicative convergent to $x \in X$.

Definition 1.7. (Ozavser. M., Cevikel. A. C. [17]). Let (X,d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0,1)$ such that $d(fx, fy) \leq d(x,y)^\lambda$ for all $x, y \in X$.

Definition 1.8. (Ozavser. M., Cevikel. A. C. [17]). Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces and $f : X \to Y$ be a function. If for every $\varepsilon > 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, then we call f multiplicative continuous at $x \in X$.

Lemma 1.9. (Ozavser. M., Cevikel. A. C. [17]). Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to x \ (n \to \infty)$ if and only if $d(x_n, x) \to 1 \ (n \to \infty)$.

Lemma 1.10. (Ozavser. M., Cevikel. A. C. [17]). Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1 \ (m,n \to \infty)$.

In 1973, Geraghty.M.A. [8] introduced an extension of the contraction in which the contraction constant was replaced by a function having some specified properties. The following notation introduced by Geraghty namely,

$$S = \{\beta : [0,\infty) \to [0,1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$$

Definition 1.11. (Geraghty.M.A. [8]). Let (X,d) be a metric space. A self map $f : X \to X$ is said to be a Geraghty contraction if there exists $\beta \in S$ such that $d(f(x), f(y)) \le \beta(d(x,y)).d(x,y) \quad \forall x, y \in X$.

Recently Nisha Sharma, Kamal Kumar and Sharma.S [15] proved the following theorem for two self mappings with rational contraction condition to get the common fixed point between the self maps.

Theorem 1.12. (Nisha Sharma et al. [15]). Let S and T be mappings of a complete multiplicative metric space (X,d) into itself satisfying the condition, $S(X)\subset X$, $T(X)\subset X$

$$d(Sx,Ty) \le \{ \max \{\frac{d(x,Sx)[d(y,Sx)+d(y,Ty)]}{1+d(Sx,Ty)}, \frac{d(y,Sx)d(x,Ty)+d(x,y)d(Sx,y)}{d(Sx,Ty)+d(Sx,y)}, \frac{d(x,Sx)d(y,Sx)+d(x,y)d(Sx,Ty)}{d(y,Ty)+d(y,Sx)}, \frac{d(y,Ty)d(x,Ty)+d(x,Ty)d(y,Sx)}{d(y,Ty)+d(y,Sx)}\}\}^{\lambda}$$

for all $x, y \in X$, where $\lambda \in (0, \frac{1}{2})$. Then S and T have unique common fixed point.

II. MAIN RESULT

In this section we mainly improve and extend theorem 1.12. Incidentally, we simplify the proof of theorem 1.12. We first give an alternate simple proof of Theorem1.12.

Theorem 2.1. Let S and T be mappings of a complete multiplicative metric space (X,d) into itself satisfying the condition $S(X)\subset X$, $T(X)\subset X$ (2.1.1)

$$d(Sx,Ty) \le \{max\{\frac{d(x,Sx)[d(y,Sx)+d(y,Ty)]}{1+d(Sx,Ty)}, \frac{d(y,Sx)d(x,Ty)+d(x,y)d(Sx,y)}{d(Sx,Ty)+d(Sx,y)}, \frac{d(x,Sx)d(y,Sx)+d(x,y)d(Sx,Ty)}{d(y,Ty)+d(y,Sx)}, \frac{d(y,Ty)d(x,Ty)+d(x,Ty)d(y,Sx)}{d(y,Ty)+d(y,Sx)}\}\}^{\lambda}$$

(2.1.2) for all $x, y \in X$, where $\lambda \in (0, \frac{1}{2})$. Then S and T have unique common fixed point.

Proof: Let x_0 be an arbitrary point in X , Since $S(X) \subseteq X$ and $T(X) \subseteq X$.

we construct the sequence $\{x_n\} in X$, such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1} \quad \forall n \ge 0$ (2.1.3)

Suppose $Sx = x,$ and $Ty = y$ then From (2.1.2)

$$d(Sx,Ty) = d(x,y) \le \{\max\{\frac{d(x,x)[d(y,x)+d(y,y)]}{1+d(x,y)}, \frac{d(y,x)d(x,y)+d(x,y)d(x,y)}{d(x,y)+d(x,y)}, \frac{d(x,x)d(y,x)+d(x,y)d(x,y)}{d(y,y)+d(y,x)}, \frac{d(y,y)d(x,y)+d(x,y)d(y,x)}{d(y,y)+d(y,x)}\}\}^{\lambda}$$

$$= \{\max\{1, d(x,y), d(x,y), d(x,y)\}\}^{\lambda}$$

$\therefore d(x,y) \le \{d(x,y)\}^{\lambda} < d(x,y)$ a contradiction if $x \ne y$ $(\because \lambda \in (0, \frac{1}{2}))$

$\therefore x = y$

$\therefore Sx = Ty.$

Suppose $Sx = x$ put $x = y$ in (2.1.2). Then

$$d(Sx,Ty) = d(x,Tx) \le$$

$$\{\max\{\frac{d(x,x)[d(x,x)+d(x,Tx)]}{1+d(x,Tx)}, \frac{d(x,x)d(x,Tx)+d(x,x)d(x,x)}{d(x,Tx)+d(x,x)}, \frac{d(x,x)d(x,x)+d(x,x)d(x,Tx)}{d(x,Tx)+d(x,x)}, \frac{d(x,Tx)d(x,Tx)+d(x,Tx)d(x,x)}{d(x,Tx)+d(x,x)}\}\}^{\lambda}$$

$$=\{\max\{1,1,1,d(x,Tx)\}\}^{\lambda}$$

$\therefore d(Sx,Ty)=d(x,Tx)\le\{d(x,Tx)\}^{\lambda}<d(x,Tx)$ a contradiction, if $x\neq Tx$

$\therefore x=Tx$ i.e., $y=Ty$

Thus the fixed points sets of S and T are the same. Let x and y be common fixed points of S and T. Then $Sx=x=Tx$ and $Sy=y=Ty \Rightarrow Sx=Ty$. Therefore S and T have unique common fixed point.

Put $y=Sx$ in (2.1.2), Then

$$d(Sx,TSx)\le\{\max\{\frac{d(x,Sx)[d(Sx,Sx)+d(Sx,TSx)]}{1+d(Sx,TSx)},\frac{1.d(x,TSx)+d(x,Sx).1}{d(Sx,TSx)+1},$$

$$\frac{d(x,Sx).1+d(x,Sx)d(Sx,TSx)}{d(Sx,TSx)+1},\frac{d(Sx,TSx)d(x,TSx)+d(x,TSx).1}{d(Sx,TSx)+1}\}\}^{\lambda}$$

$$=\{\max\{d(x,Sx),\frac{d(x,TSx)+d(x,Sx)}{1+d(Sx,TSx)},d(x,Sx),d(x,TSx)\}\}^{\lambda} \quad (2.1.4)$$

Now $\frac{d(x,TSx)+d(x,Sx)}{1+d(Sx,TSx)}=\frac{d(x,Sx)[d(Sx,TSx)+1]}{1+d(Sx,TSx)}=d(x,Sx)$

From (2.1.4) we have $d(Sx,TSx)\le\{\max\{d(x,Sx),d(x,TSx)\}\}^{\lambda}$

$$\le\{\max\{d(x,Sx),d(x,Sx).d(Sx,TSx)\}\}^{\lambda}$$

$$=\{d(x,Sx).d(Sx,TSx)\}^{\lambda}$$

$$\therefore\{d(Sx,TSx)\}^{1-\lambda}\le\{d(x,Sx)\}^{\lambda}$$

$$\therefore d(Sx,TSx)\le\{d(x,Sx)\}^{\frac{\lambda}{1-\lambda}} \quad (2.1.5)$$

Again $d(TSx,STSx)=d(STSx,TSx) \quad =d(STy,Ty)$

$$\le\{\max\{\frac{d(Ty,STy)[d(y,STy)+d(y,Ty)]}{1+d(STy,Ty)},\frac{d(y,STy)d(Ty,Ty)+d(Ty,y)d(STy,y)}{d(STy,Ty)+d(STy,y)},$$

$$\frac{d(Ty,STy)d(y,STy)+d(Ty,y)d(STy,Ty)}{d(y,Ty)+d(y,STy)},\frac{d(y,Ty)d(Ty,Ty)+d(Ty,Ty)d(y,STy)}{d(y,Ty)+d(y,STy)}\}\}^{\lambda} \quad (2.1.6)$$

Now by the above inequality,

$$\frac{d(Ty,STy)[d(y,STy)+d(y,Ty)]}{1+d(STy,Ty)}\le\frac{d(Ty,STy)[d(y,Ty).d(Ty,STy)+d(y,Ty)]}{1+d(STy,Ty)}=d(Ty,STy).d(y,Ty)$$

$$\frac{d(y,STy)d(Ty,Ty)+d(Ty,y)d(STy,Ty)]}{d(STy,Ty)+d(STy,Ty)}\le\frac{d(y,Ty)d(Ty,STy).1+d(Ty,y)d(STy,Ty)]}{d(STy,Ty)+d(STy,Ty)}$$

$$\le\frac{d(y,Ty)[d(Ty,STy)+d(STy,Ty)]}{[d(STy,Ty)+d(STy,Ty)]}=d(y,Ty)$$

$$\frac{d(Ty,STy)d(y,STy)+d(Ty,y)d(STy,Ty)]}{d(y,Ty)+d(y,STy)}=\frac{d(Ty,STy)[d(y,STy)+d(Ty,y)]}{[d(y,Ty)+d(y,STy)]}=d(Ty,STy)$$

And $\frac{d(y,Ty)d(Ty,Ty)+d(Ty,Ty)d(y,STy)}{d(y,Ty)+d(y,STy)}=\frac{d(y,Ty).1+1.d(y,STy)}{d(y,Ty)+d(y,STy)}=1$

Therefore $From(2.1.6)$, $d(STy,Ty)\le\{\max\{d(Ty,STy).d(y,Ty),d(y,Ty),d(Ty,STy),1\}\}^{\lambda}$

i.e., $d(STy,Ty)\le\{d(Ty,STy).d(y,Ty)\}^{\lambda}$

i.e., $d(STy,Ty)\le\{d(y,Ty)\}^{\frac{\lambda}{1-\lambda}}=\{d(Ty,y)\}^{\frac{\lambda}{1-\lambda}}$

$$\therefore d(STSx,TSx)\le\{d(Ty,y)\}^{\frac{\lambda}{1-\lambda}}$$
$$=\{d(TSx,Sx)\}^{\frac{\lambda}{1-\lambda}}$$
$$=\{d(Sx,TSx)\}^{\frac{\lambda}{1-\lambda}}$$
$$=[\{d(x,Sx)\}^{\frac{\lambda}{1-\lambda}}]^{\frac{\lambda}{1-\lambda}} \qquad \text{(from(2.1.5))}$$
$$=\{d(x,Sx)\}^{(\frac{\lambda}{1-\lambda})^2}.$$

By induction $d(x_2,x_1)\le\{d(x_1,x_0)\}^{\frac{\lambda}{1-\lambda}}$

$$d(x_3,x_2)\le\{d(x_2,x_1)\}^{\frac{\lambda}{1-\lambda}}$$
$$\le\{d(x_1,x_0)\}^{(\frac{\lambda}{1-\lambda})^2}.$$

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In general $d(x_{n+1},x_n)\le\{d(x_n,x_{n-1})\}^{\frac{\lambda}{1-\lambda}}$

$$\le\ldots\ldots\ldots\ldots$$
$$\le\ldots\ldots\ldots\ldots$$
$$\le\{d(x_1,x_0)\}^{\left(\frac{\lambda}{1-\lambda}\right)^n}$$

For $n\in N$,

$$d(x_n,x_{n+k})\le d(x_n,x_{n+1}).d(x_{n+1},x_{n+2})\ldots\ldots\ldots d(x_{n+k-1},x_{n+k})$$
$$\le\{d(x_1,x_0)\}^{\left(\frac{\lambda}{1-\lambda}\right)^n}.\{d(x_1,x_0)\}^{\left(\frac{\lambda}{1-\lambda}\right)^{n+1}}\ldots\ldots\ldots\{d(x_1,x_0)\}^{\left(\frac{\lambda}{1-\lambda}\right)^{n+k+1}}$$
$$=\{d(x_1,x_0)\}^{h^n+h^{n+1}+\ldots\ldots+h^{n+k-1}} \qquad (write\ \frac{\lambda}{1-\lambda}=h)$$
$$=\{d(x_1,x_0)\}^{\frac{h^n}{1-h}}\to 1 \quad as\, n,k\to\infty$$

Therefore $\{x_n\}$ is a Multiplicative Cauchy sequence in X.

Since X is a complete multiplicative metric space, there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$

Consequently, the subsequence $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ of $\{x_n\}$ also converge to the point $x^*\in X$.

Now $d(Sx_{2n},Tx^*)\le\{\max\{\frac{d(x_{2n},Sx_{2n})[d(x^*,Sx_{2n})+d(x^*,Tx^*)]}{1+d(Sx_{2n},Tx^*)},$

$$\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tx^*)+d(x_{2n},x^*)d(Sx_{2n},x^*)}{d(Sx_{2n},Tx^*)+d(Sx_{2n},x^*)},$$
$$\frac{d(x_{2n},Sx_{2n})d(x^*,Sx_{2n})+d(x_{2n},x^*)d(Sx_{2n},Tx^*)}{d(x^*,Tx^*)+d(x^*,Sx_{2n})},$$
$$\frac{d(x^*,Tx^*)d(x_{2n},Tx^*)+d(x_{2n},Tx^*)d(x^*,Sx_{2n})}{d(x^*,Tx^*)+d(x^*,Sx_{2n})}\}\}^{\lambda}$$

Suppose $Tx^*\ne x^*$, On letting $n\to\infty$

Then

$$d(x^*,Tx^*)\le \{\max\{\frac{1+d(x^*,Tx^*)}{1+d(x^*,Tx^*)},\ \frac{d(x^*,Tx^*)+1}{d(x^*,Tx^*)+1},$$
$$\frac{1+d(x^*,Tx^*)}{d(x^*,Tx^*)+1},\ \frac{d(x^*,Tx^*)d(x^*,Tx^*)+d(x^*,Tx^*)}{d(x^*,Tx^*)}\}\}^{\lambda}$$
$$=\{\ \max\ \{1,1,1,d(x^*,Tx^*)\}\}^{\lambda}$$

$\therefore d(x^*,Tx^*)\le\{d(x^*,Tx^*)\}^{\lambda}<d(x^*,Tx^*)$ a contradiction, if $x^*\ne Tx^*$

$\therefore Tx^*=x^*$

Similarly we can show that $d(Sx^*,x^*)\le\{d(Sx^*,x^*)\}^{\lambda}<d(Sx^*,x^*)$ a contradiction, if $x^*\ne Sx^*$

$\therefore Sx^*=x^*$

$\therefore x^*$ is a common fixed point of S and T.

Therefore $Sx^*=Tx^*=x^*$ (2.1.7)

Uniqueness: Let x^* and y^* be two common fixed points of S and T

Suppose $x^*\ne y^*$ we have

$$d(x^*,y^*)=d(Sx^*,Ty^*)\le\{\ \max\ \{\frac{d(x^*,Sx^*)[d(y^*,Sx^*)+d(y^*,Ty^*)]}{1+d(Sx^*,Ty^*)},$$
$$\frac{d(y^*,Sx^*)d(x^*,Ty^*)+d(x^*,y^*)d(Sx^*,y^*)}{d(Sx^*,Ty^*)+d(Sx^*,y^*)},$$
$$\frac{d(x^*,Sx^*)d(y^*,Sx^*)+d(x^*,y^*)d(Sx^*,Ty^*)}{d(y^*,Ty^*)+d(y^*,Sx^*)},$$
$$\frac{d(y^*,Ty^*)d(x^*,Ty^*)+d(x^*,Ty^*)d(y^*,Sx^*)}{d(y^*,Ty^*)+d(y^*,Sx^*)}\}\}^{\lambda}$$
$$\le\{\ \max\ \{\frac{d(x^*,x^*)[d(y^*,x^*)+d(y^*,y^*)]}{1+d(x^*,y^*)},$$
$$\frac{d(y^*,x^*)d(x^*,y^*)+d(x^*,y^*)d(x^*,y^*)}{d(x^*,y^*)+d(x^*,y^*)},$$
$$\frac{d(x^*,x^*)d(y^*,x^*)+d(x^*,y^*)d(x^*,y^*)}{d(y^*,y^*)+d(y^*,x^*)},$$
$$\frac{d(y^*,x^*)d(x^*,y^*)+d(x^*,y^*)d(y^*,x^*)}{d(y^*,y^*)+d(y^*,x^*)}\}\}^{\lambda}$$
$$\le\{\ \max\ \{1,d(x^*,y^*),d(x^*,y^*),d(x^*,y^*)\}\}^{\lambda}$$

$\therefore d(x^*,y^*)\le\{d(x^*,y^*)\}^{\lambda}<d(x^*,y^*)$ a contradiction

$\therefore x^*=y^*$.

In the following we introduce multiplicative Geraghty function and the notion of multiplicative $\psi-\beta$ Geraghty Contraction.

Definition 2.2. A function $\beta:(1,\infty)\to(0,1)$ is called multiplicative Geraghty function if $\beta(t_n)\to1\Rightarrow t_n\to1$ as $n\to\infty$.

Example 2.3. Define $\beta:(1,\infty)\to(0,1)$ by $\beta(t)=\frac{1}{t}$

Notation:
$Let\ \Psi = \{\psi : [1,\infty) \to [1,\infty) / \psi\ is\ increasing,\ cotinuous,\ \ \psi(t) = 1\ iff\ t = 1\ and\ \psi(t) < t\ if\ t > 1\}$.

Example 2.4. Define $\psi(t) = t^{\frac{1}{2}}$ on $[1,\infty)$ then $\psi \in \Psi$.

Definition 2.5. Suppose (X,d) is a multiplicative metric space, β is a multiplicative Geraghty function, $\psi \in \Psi$. Suppose $f : X \to X$ is such that $\psi(d(fx, fy)) \le \beta(M(x,y)).\psi(M(x,y))$, where $M(x,y)$ is a function of x, y involving d. Then we say that f is a multiplicative $\psi - \beta$ Geraghty Contraction.

Now we state and prove our main theorem.

Theorem 2.6. Let S and T be mappings of a complete multiplicative metric space (X,d) into itself satisfying the condition $S(X) \subset X$, $T(X) \subset X$

$$\psi[d(Sx,Ty)] \le \beta[\psi(M(x,y))].\psi[M(x,y)] \tag{2.6.1}$$

where

$$M(x,y) = \{\max\{\frac{d(x,Sx)[d(y,Sx)+d(y,Ty)]}{1+d(Sx,Ty)}, \frac{d(y,Sx)d(x,Ty)+d(x,y)d(Sx,y)}{d(Sx,Ty)+d(Sx,y)}, \frac{d(x,Sx)d(y,Sx)+d(x,y)d(Sx,Ty)}{d(y,Ty)+d(y,Sx)}, \frac{d(y,Ty)d(x,Ty)+d(x,Ty)d(y,Sx)}{d(y,Ty)+d(y,Sx)}\}\}^{\lambda} \tag{2.6.2}$$

$forall\ x, y \in X$, $\lambda \in (0, \frac{1}{2})$, and $\psi : [1,\infty) \to [1,\infty)$ is a monotonic increasing function such that $\psi(t) = 1$ iff $t = 1$ and $\psi(t) < t,\ \ \forall t > 1$ and $\beta : (1,\infty) \to (0,1), \beta(t_n) \to 1 \Rightarrow t_n \to 1$. Then S and T have unique common fixed point.

Proof: Suppose $Sx = x$. Put $x = y$ in (2.6.1). Then

$$\psi[d(x,Tx)] = \psi[d(Sx,Tx)] \le \beta[\psi(M(x,x))].\psi[M(x,x)] \tag{2.6.3}$$

where

$$M(x,x) = \{\max\{\frac{d(x,x)[d(x,x)+d(x,Tx)]}{1+d(x,Tx)}, \frac{d(x,x)d(x,Tx)+d(x,x)d(x,x)}{d(x,Tx)+d(x,x)}, \frac{d(x,x)d(x,x)+d(x,x)d(x,Tx)}{d(x,Tx)+d(x,x)}, \frac{d(x,Tx)d(x,Tx)+d(x,Tx)d(x,x)}{d(x,Tx)+d(x,x)}\}\}^{\lambda}$$

$$= \{\ \max\ \{1,1,1,d(x,Tx)\}\}^{\lambda}$$

$$= \{d(x,Tx)\}^{\lambda}$$

From (2.6.3), $\psi[d(x,Tx)] \le \beta[\psi(M(x,x))].\psi[\{d(x,Tx)^{\lambda}]$

$$< \psi[\{d(x,Tx)\}^{\lambda}]$$

Since ψ is monotonically increasing, $d(x,Tx) < \{d(x,Tx)\}^{\lambda} < d(x,Tx)$ a contradiction if $x \ne Tx$

$\therefore x = Tx$. Therefore x is a fixed point of T.

Similarly we can show that $Ty = y \Rightarrow Sy = y.$

Thus S and T have the same fixed point sets.

Suppose $y = Sx$

From (2.6.1), $\psi[d(Sx,Ty)] = \psi[d(Sx,TSx)] \le \beta[\psi(M(x,Sx))].\psi[M(x,Sx)]$ (2.6.4) where

$$M(x,Sx)=\{\max\{\frac{d(x,Sx)[d(Sx,Sx)+d(Sx,TSx)]}{1+d(Sx,TSx)},\frac{d(Sx,Sx)d(x,TSx)+d(x,Sx)d(Sx,Sx)}{d(Sx,TSx)+d(Sx,x)},$$

$$\frac{d(x,Sx)d(Sx,Sx)+d(x,Sx)d(Sx,STx)}{d(Sx,TSx)+d(Sx,Sx)},\frac{d(Sx,TSx).d(x,TSx)+d(x,TSx)d(Sx,Sx)}{d(Sx,TSx)+d(Sx,Sx)}\}\}^{\lambda}$$

$$=\{\max\{d(x,Sx),\ \frac{d(x,TSx)+d(x,Sx)]}{1+d(Sx,TSx)},d(x,Sx),d(x,TSx)\}\}^{\lambda}$$

$$Now\ \frac{d(x,TSx)+d(x,Sx)}{1+d(Sx,TSx)}\leq\ \frac{d(x,Sx).d(Sx,TSx)+d(x,Sx)}{1+d(Sx,TSx)}=\frac{d(x,Sx)[d(Sx,TSx)+1]}{1+d(Sx,TSx)}=d(x,Sx)$$

$$\therefore M(x,Sx)\leq\{\max\{d(x,Sx),d(x,TSx)\}\}^{\lambda}$$

$$\therefore M(x,Sx)=\{\max\{d(x,Sx),d(x,Sx).d(Sx,TSx)\}\}^{\lambda}=\{d(x,Sx).d(Sx,TSx)\}^{\lambda}$$

From (2.6.4), $\psi[d(Sx,TSx)]\leq\beta[\psi(M(x,Sx))].\psi[M(x,Sx)]$

$$\leq\beta[\psi(M(x,Sx))].\psi[\{d(x,Sx).d(Sx,TSx)\}^{\lambda}] \qquad (2.6.5)$$

Again suppose $x=Ty$. Then we have

$$\psi[d(TSx,STSx)]=\psi[d(Ty,STy)]=\psi[d(STy,Ty)]\leq\beta[\psi(M(Ty,y))].\psi[M(Ty,y)] \qquad (2.6.6)$$

Where

$$M(Ty,y)=\{\max\{\frac{d(Ty,STy)[d(y,STy)+d(y,Ty)]}{1+d(STy,Ty)},\frac{d(y,STy)d(Ty,Ty)+d(Ty,y)d(STy,y)}{d(STy,Ty)+d(STy,y)},$$

$$\frac{d(Ty,STy)d(y,STy)+d(Ty,y)d(STy,Ty)}{d(y,Ty)+d(y,STy)},\frac{d(y,Ty)d(Ty,Ty)+d(Ty,Ty)d(y,STy)}{d(y,Ty)+d(y,STy)}\}\}^{\lambda}$$

Now $\frac{d(Ty,STy)[d(y,STy)+d(y,Ty)]}{1+d(STy,Ty)}\leq\frac{d(Ty,STy)[d(y,Ty).d(Ty,STy)+d(y,Ty)]}{1+d(STy,Ty)}$

$$=\frac{d(Ty,STy).d(y,Ty)[d(STy,Ty)+1]}{1+d(STy,Ty)}=d(Ty,STy).d(y,Ty)$$

$$\frac{d(y,STy)d(Ty,Ty)+d(Ty,y)d(STy,y)}{d(STy,Ty)+d(STy,y)}\leq\frac{d(y,Ty).d(Ty,STy)d(Ty,Ty)+d(Ty,y)d(STy,y)}{d(STy,Ty)+d(STy,y)}$$

$$=\frac{d(y,Ty)[d(Ty,STy)+d(y,STy)]}{d(Ty,STy)+d(y,STy)}=d(y,Ty)$$

$$\frac{d(Ty,STy)d(y,STy)+d(Ty,y)d(STy,Ty)}{d(y,Ty)+d(y,STy)}=\frac{d(Ty,STy)[d(y,STy)+d(Ty,y)}{d(y,Ty)+d(y,STy)}=d(Ty,STy)$$

$$\frac{d(y,Ty)d(Ty,Ty)+d(Ty,Ty)d(y,STy)}{d(y,Ty)+d(y,STy)}=\frac{d(y,Ty).1+1.d(y,STy)}{d(y,Ty)+d(y,STy)}=1$$

$$\therefore M(Ty,y)=\{\max\{d(Ty,STy).d(y,Ty),d(y,Ty),d(Ty,STy)\}\}^{\lambda}=\{d(Ty,STy).d(y,Ty)\}^{\lambda}$$

from (2.6.6), $\psi[d(STy,Ty)]=\psi[d(STSx,TSx)]\leq\beta[\psi(M(Ty,y))].\psi[\{d(Ty,y).d(STy,Ty)\}^{\lambda}]$
(2.6.7)

Let $x_0\in X$, $x_1=Sx_0$, $x_2=Tx_1$ *and ingeneral* $Sx_{2n}=x_{n+1}$ and $Tx_{2n+1}=x_{n+2}\ \forall n\geq 0$.

Put $x=x_0$ in (2.6.5). Then

$$\psi[d(Sx_0,TSx_0)]\leq\beta[\psi(M(x_0,Sx_0))].\psi[M(x_0,Sx_0)]$$

i.e., $\psi[d(x_1, x_2)] \le \beta[\psi(M(x_0, x_1))].\psi[\{d(x_0, x_1).d(x_1, x_2)\}^{\lambda}]$

$\psi[d(x_2, x_1) \le \beta[\psi(M(x_1, x_0))].\psi[\{d(x_0, x_1).d(x_1, x_2)\}^{\lambda}]$

Put $y = Sx_0 = x_1$. Then from (2.6.7) we get

$\psi[d(x_3, x_2) \le \beta[\psi(M(x_2, x_1))].\psi[\{d(x_1, x_2).d(x_2, x_3)\}^{\lambda}]$

By induction, $\psi[d(x_{n+2}, x_{n+1}) \le \beta[\psi(M(x_{n+1}, x_n))].\psi[\{d(x_{n+1}, x_{n+2}).d(x_n, x_{n+1})\}^{\lambda}]$ (2.6.8)

$< \psi[\{d(x_{n+2}, x_{n+1}).d(x_{n+1}, x_n)\}^{\lambda}]$

Suppose $d(x_{n+1}, x_n) \le d(x_{n+2}, x_{n+1})$

Then

$\psi(d(x_{n+2}, x_{n+1})) < \psi[\{(d(x_{n+2}, x_{n+1})^2\}^{\lambda}]$

$= \psi[\{(d(x_{n+2}, x_{n+1})^{2\lambda}\}]$

Since ψ *is* monotonically increasing,

$d(x_{n+2}, x_{n+1}) < d(x_{n+2}, x_{n+1})^{2\lambda} < d(x_{n+2}, x_{n+1}),$ *a contradiction.*

Therefore $d(x_{n+2}, x_{n+1}) < d(x_{n+1}, x_n)^{2\lambda}$ (2.6.9)

And hence $d(x_{n+2}, x_{n+1}) < d(x_{n+1}, x_n)$

Now we show that $d(x_{n+2}, x_{n+1})$ is strictly decreasing sequence.

Suppose $d(x_{n+1}, x_n)$ is decreasing to $r \ge 1.$

Then, on letting $n \to \infty$, from (2.6.9) we get $r \le r^{2\lambda}$

$\therefore r = 1.$

Now we show that $\{x_n\}$ is a multiplicative Cauchy sequence.

Write $d(x_{n+1}, x_n) = t_n.$

$For\, n, m \in N, n < m$

$$d(x_n, x_m) \le d(x_n, x_{n+1}).d(x_{n+1}, x_{n+2})............d(x_{m-1n}, x_m)$$

$$= t_n.t_{n+1}.t_{n+2}............t_{m-1}$$

$$< t_n.(t_n)^{2\lambda}.(t_n)^{2\lambda^2}............(t_n)^{2\lambda^{m-n-1}}$$

$$= (t_n)^{[1+s+s^2+............+s^{m-n-1}]} \qquad (write\ s = 2\lambda)$$

$$\le (t_n)^{\frac{s}{1-s}} \to 1 \ \ (\sin ce\, t_n \to 1 as\, n \to \infty\, and\, \frac{s}{1-s} < 1)$$

$\therefore d(x_n, x_m) \to 1 \ \ (as\ m, n \to \infty).$ Therefore $\{x_n\}$ is a multiplicative Cauchy sequence in X.

Since X is a complete multiplicative metric space , there exists a point $x^* \in X$, such that $x_n \to x^*$. More over the sub sequences $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ also converge to x^*.

Now we show that $Sx^* = x^*$.

Now $\psi[d(Sx^*, x_{2n+2})] = \psi[d(Sx^*, Tx_{2n+1})] \le \beta[\psi(M(x^*, x_{2n+1})].\psi[M(x^*, x_{2n+1})]).$ (2.6.10)

Where

$$M(x^*, x_{2n+1}) = \{\max\{\frac{d(x^*, Sx^*)[d(x_{2n+1}, Sx^*) + d(x_{2n+1}, Tx_{2n+1})]}{1 + d(Sx^*, Tx_{2n+1})},$$

$$\frac{d(x_{2n+1}, Sx^*)d(x^*, Tx_{2n+1}) + d(x^*, x_{2n+1})d(Sx^*, x_{2n+1})}{d(Sx^*, Tx_{2n+1}) + d(Sx^*, x_{2n+1})},$$

$$\frac{d(x_*,Sx^*)d(x_{2n+1},Sx^*,)+d(x^*,x_{2n+1})d(Sx^*,Tx_{2n+1})}{d(x_{2n+1},Tx_{2n+1})+d(x_{2n+1},Sx^*)},$$
$$\frac{d(x_{2n+1},Tx_{2n+1})d(x^*,Tx_{2n+1})+d(x^*,Tx_{2n+1})d(x_{2n+1},Sx^*)}{d(x_{2n+1},Tx_{2n+1})+d(x_{2n+1},Sx^*)}\}\}^{\lambda}$$

We observe that

$$\frac{d(x_*,Sx^*)[d(x_{2n+1},Sx^*)+d(x_{2n+1},Tx_{2n+1})]}{1+d(Sx^*,Tx_{2n+1})} \to \frac{d(x^*,Sx^*)[d(x^*,Sx^*)+d(x^*,x^*)]}{1+d(Sx^*,x^*)}$$

as $n\to\infty$ $\quad = d(Sx^*,x^*)$.

$$\frac{d(x_{2n+1},Sx^*)d(x^*,Tx_{2n+1})+d(x^*,x_{2n+1})d(Sx^*,x_{2n+1})}{d(Sx^*,Tx_{2n+1})+d(Sx^*,x_{2n+1})} \to \frac{d(x^*,Sx^*)d(x^*,x^*)+d(x^*,x^*)d(Sx^*,x^*)}{d(Sx^*,x^*)+d(Sx^*,x^*)}$$

as $n\to\infty$ $\quad =1$

$$\frac{d(x_*,Sx^*)d(x_{2n+1},Sx^*,)+d(x^*,x_{2n+1})d(Sx^*,Tx_{2n+1})}{d(x_{2n+1},Tx_{2n+1})+d(x_{2n+1},Sx^*)} \to$$
$$\frac{d(x^*,Sx^*)d(x^*,Sx^*)+d(x^*,x^*)d(x^*,Sx^*)}{d(x^*,x^*)+d(x^*,Sx^*)} \quad \text{as} \quad n\to\infty$$

$$= d(Sx^*,x^*).$$

$$\frac{d(x_{2n+1},Tx_{2n+1})d(x^*,Tx_{2n+1})+d(x^*,Tx_{2n+1})d(x_{2n+1},Sx^*)}{d(x_{2n+1},Tx_{2n+1})+d(x_{2n+1},Sx^*)} \to$$
$$\frac{d(x^*,x^*)d(x^*,x^*)+d(x^*,x^*)d(x^*,Sx^*)}{d(x^*,x^*)+d(x^*,Sx^*)} \quad \text{as} \quad n\to\infty$$

$$=1$$

$\therefore M(x^*,x_{2n+1})\to\{\max\{d(Sx^*,x^*),1,d(Sx^*,x^*),1\}^{\lambda}\}=\{d(Sx^*,x^*)\}^{\lambda}$ as $n\to\infty$

Write $t_n=\psi(M(x^*,x_{2n+1}))$. Then

<u>*Case (i)*</u>: $\beta(t_n)\to 1$. Then $t_n\to 1$ $\qquad$ *(by the property of* β*)*

$$\Rightarrow \psi(M(x^*,x_{2n+1}))\to 1.$$

On letting $n\to\infty$, *from* (2.6.10) *we get*)

$\psi\{d(Sx^*,x^*)\}\le 1.$

Therefore $\psi(d(Sx^*,x^*))=1.$

Therefore $d(Sx^*,x^*)=1.$

Therefore $Sx^*=x^*$

i.e., x^* is a fixed point of S.

<u>*Case (ii)*</u>: Suppose $\beta(t_n)$ does not converge to 1.

Then we may suppose without loss of generality, that there exists $\alpha<1$ such that $\beta(t_n)<\alpha<1\ \forall n$.

From (2.6.10), we have

$$\psi\{d(Sx^*,x_{2n+1})\}\le\beta(t_n).(t_n)$$
$$<\alpha.(t_n)$$

On letting $n\to\infty$, we get

$\psi(d(Sx^*,x^*))\le\alpha.\psi(d(Sx^*,x^*)^{\lambda})<\psi(d(Sx^*,x^*)^{\lambda})\le\psi(d(Sx^*,x^*)),$ a contradiction.

Hence Case (ii) does not arise. Therefore $\beta(t_n) \to 1$.

Therefore by case(i), $Sx^* = x^*$

Therefore x^* is a fixed point of S and hence is a fixed point of T.

$\underline{Uniqueness}$: Let x^* and y^* be two common fixed points of S and T.

We prove that $x^* = y^*$.

Now, $M(x^*, y^*) = \{ \max\{\frac{d(x^*,Sx^*)[d(y^*,Sx^*)+d(y^*,Ty^*)]}{1+d(Sx^*,Ty^*)},$

$$\frac{d(y^*,Sx^*)d(x^*,Ty^*)+d(x^*,y^*)d(Sx^*,y^*)}{d(Sx^*,Ty^*)+d(Sx^*,y^*)},$$

$$\frac{d(x^*,Sx^*)d(y^*,Sx^*)+d(x^*,y^*)d(Sx^*,Ty^*)}{d(y^*,Ty^*)+d(y^*,Sx^*)},$$

$$\frac{d(y^*,Tx^*)d(x^*,Ty^*)+d(x^*,Ty^*)d(y^*,Sx^*)}{d(y^*,Ty^*)+d(y^*,Sx^*)}\}\}^{\lambda}$$

$$= \{ \max \{1, d(x^*, y^*), d(x^*, y^*), d(x^*, y^*)\}\}^{\lambda}$$

$$= \{d(x^*, y^*)\}^{\lambda}$$

$$\therefore \psi[d(x^*, y^*)] \le \beta[\psi(M(x^*, y^*)].\psi[M(x^*, y^*)].$$

$$= \beta[\psi(M(x^*, y^*)].\psi[d(x^*, y^*)^{\lambda}].$$

$\psi[d(x^*, y^*)] < \psi[d(x^*, y^*)^{\lambda}]$, if $x^* \ne y^*$.

Since ψ is monotonically increasing, $d(x^*, y^*) < d(x^*, y^*)^{\lambda}$, a contradiction if $x^* \ne y^*$.

Therefore $x^* = y^*$

$\therefore S$ and T have unique common fixed point x^*.

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