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A Nonlinear Optimization Problemsubjected To Hamacher-FRE Restrictions

Amin Ghodousian[1]

Faculty of Engineering Science, College of Engineering, University of Tehran, P.O.Box 11365-4563, Tehran, Iran.

Corresponding Author : Amin Ghodousian

ABSTRACT:*Hamacher family of t-norms is a parametric family of continuous strict t-norms, whose members are decreasing functions of the parameter. In this paper, we study a nonlinear optimization problem constrained byspecial system of fuzzy relational equations(FRE)in which fuzzy t-norms are considered as the members of the Hamacher family. The resolution of the feasible solutions setis firstly investigated when it is defined with max-Hamacher composition.Also, some necessary and sufficient conditions are presented for determining the feasibility and some procedures are proposed for simplifying the problem. Based on thetheoretical properties of the problem, a genetic algorithm is used, which preserves the feasibility of new generated solutions. Moreover, a method is presented to generate feasible max-Hamacher FREs as test problems for evaluating the performance of our algorithm. The proposed method has been compared with Lu and Fang's algorithm. The obtained results confirm the high performance of the proposed method in solving such nonlinear problems.*

KEYWORDS: *Fuzzy relational equations, nonlinear optimization, genetic algorithm.*

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I. INTRODUCTION

In this paper, we study the following nonlinear problem in which the constraints are formed as fuzzy relational equations defined by Hamacher t-norm:

$$\begin{aligned} \min \quad & f(x) \\ & A\varphi x = b \quad (1) \\ & x \in [0,1]^n \end{aligned}$$

where $I = \{1,2,...,m\}$, $J = \{1,2,...,n\}$, $A = (a_{ij})_{m\times n}$, $0 \le a_{ij} \le 1$ ($\forall i \in I$ and $\forall j \in J$), is a fuzzy matrix, $b = (b_i)_{m\times 1}, 0 \le b_i \le 1$($\forall i \in I$), is an m–dimensional fuzzy vector, and "φ" is the max-Hamacher composition, that is,

$$\varphi(x,y) = T_H^{\alpha}(x,y) = \begin{cases} 0 & \alpha = x = y = 0 \\ \dfrac{xy}{\alpha + (1-\alpha)(x+y-xy)} & otherwise \end{cases}$$

in which $\alpha \ge 0$.

If a_i is the i'th row of matrix A, then problem (1) can be expressed as follows:

$$\begin{aligned} \min \quad & f(x) \\ & \varphi(a_i, x) = b_i \quad , \; i \in I \\ & x \in [0,1]^n \end{aligned}$$

where the constraints mean:

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[1] . Email address: a.ghodousian@ut.ac.ir

$$\varphi(a_i,x)=\max_{j\in J}\{\varphi(a_{ij},x_j)\}=\max_{j\in J}\{T_H^{\alpha}(a_{ij},x_j)\}=b_i \quad ,\forall i\in I$$

and

$$T_H^{\alpha}(a_{ij},x_j)=\begin{cases}0 & \alpha=a_{ij}=x_j=0\\ \dfrac{a_{ij}x_j}{\alpha+(1-\alpha)(a_{ij}+x_j-a_{ij}x_j)} & otherwise\end{cases}$$

As mentioned, Members of the Hamacher family of t-norms are decreasing functions of the parameter α and each member of this familyis actually a strict t-norm [8]. In [44] some new operational rules of hesitant fuzzy sets were introduced based on the Hamacher t-norm and t-conorm, in which a family of hesitant fuzzy Hamacher operators was proposed for aggregating hesitant fuzzy information. In [45], the mono-tonicity of alternative scores derived from Hamacher arithmetic and geometric aggregation operators. They also investigated the relationship between alternative scores generated by Hamacher arithmetic and geometric aggregation operators. In [46], the authors focused on examining the general parametric Hamacher t-norm, where the free parameter quite essentially influences the quality of modeling and the learning capability of the model identification system.

The theoryof fuzzy relational equations was firstly proposed by Sanchez [55].He introduced a FRE whit max-min composition and applied the model to medical diagnosis in Brouwerian logic.Nowadays, it is well-known that many issues associated with a body knowledge can be treated as FRE problems [47]. In addition to such applications, FRE theory has been applied in many fields including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, and so on. Pedrycz [48] categorized and extended two ways of the generalizations of FRE in terms of sets under discussion and various operations which are taken into account. Since then, many theoretical improvements have been investigated and many applications have been presented [5,11,24,28,32,41,49,51,52,60,62,68].

The solvability and the finding of solutions set are the primary (and the most fundamental) subject concerning FRE problems. Many studies have reported fuzzy relational equations with max-min and max-product compositions. Both compositions are special cases of the max-triangular-norm (max-t-norm) [2,3,37,38,43]. It is well-known that the solution set of FRE (if it is nonempty) defined by continuous max-t-norm composition is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [6]. Lin et al. [38] demonstrated that all systems of max-continuous t-norm fuzzy relational equations, for example, max-product, max-continuous Archimedean t-norm and max-arithmetic mean are essentially equivalent, because they are all equivalent to the set covering problem. Over the last decades, the solvability of FRE defined with different max-t compositions has been investigated by many researches [50,53,54,56,58,59,63,67,71]. It is worth to mention that Li and Fang [36] provided a complete survey and a detailed discussion on fuzzy relational equations. They studied the relationship among generalized logical operators involved in the construction of FRE and introduced the classification of basic fuzzy relational equations.

Optimizing an objective function subjected to a system of fuzzy relational equations or inequalities (FRI)is one of the most interesting and on-going topics among the problems related to the FRE (or FRI) theory [1,9,19-27,31,34,39,57,64,69]. By far the most frequently studied aspect is the determination of a minimizer of a linear objective function and the use of the max-min composition [1,20].So, it is an almost standard approach to translate this type of problem into a corresponding 0-1 integer linear programming problem, which is then solved using a branch and bound method [10,65]. In [33] an application of optimizing the linear objective with max-min composition was employed for the streaming media provider seeking a minimum cost while fulfilling the requirements assumed by a three-tier framework. Chang and Shieh [1] presented new theoretical results concerning the linear optimization problem constrained by fuzzy max-min relation equations by improving an upper bound on the optimal objective value. The topic of the linear optimization problem was also investigated with max-product operation [19,26,40]. Loetamonphong and Fang defined two sub-problems by separating negative and non-negative coefficients in the objective function and then obtained the optimal solution by combining those of the two sub-problems [40]. Also, in [26] and [19], some necessary conditions of the feasibility and simplification techniques were presented for solving FRE with max-product composition. Moreover, some studies have determined a more general operator of linear optimization with replacement of max-min and max-product compositions with a max-t-norm composition [17,25,34,57], max-average composition [31,64] or max-star composition [22].

Recently, many interesting generalizations of the linear and non-linear programming problems constrained by FRE or FRI have been introduced and developed based on composite operations and fuzzy relations used in the definition of the constraints, and some developments on the objective function of the problems [4,7,12,20,35,39,66].For instance, the linear optimization of bipolar FRE was studied by some researchers where FRE was defined with max-min composition [12] and max-Lukasiewicz composition [35,39]. In [35] the authors introduced the optimization problem subjected to a system of bipolar FRE defined as $X(A^+,A^-,b)=\{x\in[0,1]^m : x\circ A^+ \vee \tilde{x}\circ A^- = b\}$ where $\tilde{x}_i = 1-x_i$ for each component of $\tilde{x}=(\tilde{x}_i)_{1\times m}$ and the notations '' $\vee$ ''' and '' $\circ$ '' denote max operation and the max-Lukasiewicz composition, respectively. They translated the problem into a 0-1 integer linear programming problem which is then solved using well-developed techniques. In [39], the foregoing problem was solved by an analytical method based on the resolution and some structural properties of the feasible region (using a necessary condition for characterizing an optimal solution and a simplification process for reducing the problem). In [21] the authors focused on the algebraic structure of two fuzzy relational inequalities $A\varphi x \le b^1$ and $D\varphi x \ge b^2$, and studied a mixed fuzzy system formed by the two preceding FRIs, where φ is an operator with (closed) convex solutions. Yang [70] studied the optimal solution of minimizing a linear objective function subject to fuzzy relational inequalities where the constraints defined as $a_{i1}\wedge x_1 + a_{i2}\wedge x_2 + ... + a_{in}\wedge x_n \ge b_i$ for $i=1,...,m$ and $a\wedge b=\min\{a,b\}$. He presented an algorithm based on some properties of the minimal solutions of the FRI.Ghodousianet al. [16,20] introduced FRI-FC problem $\min\{\, c^T x : A\varphi x \circ b ,\ x\in[0,1]^n \}$, where φ is max-min composition and " $\circ$ " denotes the relaxed or fuzzy version of the ordinary inequality "$\le$".

Another interesting generalizations of such optimization problems are related to objective function. Wu et al. [66] represented an efficient method to optimize a linear fractional programming problem under FRE with max-Archimedean t-norm composition. Dempe and Ruziyeva [4] generalized the fuzzy linear optimization problem by considering fuzzy coefficients. Dubey et al. studied linear programming problems involving interval uncertainty modeled using intuitionistic fuzzy set [7]. If the objective function is $z(x)=\max_{i=1}^{n}\{\min\{c_i,x_i\}\}$ with $c_i\in[0,1]$, the model is called the latticized problem [61]. Also, Yang et al. [69] introduced another version of the latticized programming problem subject to max-prod fuzzy relation inequalities with application in the optimization management model of wireless communication emission base stations. The latticized problem was defined by minimizing objective function $z(x)=x_1\vee x_1\vee ...\vee x_n$ subject to feasible region $X(A,b)=\{x\in[0,1]^n : A\circ x\ge b\}$ where '' $\circ$ '' denotes fuzzy max-product composition.They also presented an algorithm based on the resolution of the feasible region. On the other hand, Lu and Fang considered the single non-linear objective function and solved it with FRE constraints and max-min operator [42]. They proposed a genetic algorithm for solving the problem.Hassanzadeh et al. [29] used the same GA proposed by Lu and Fang to solve a similar nonlinear problem constrained by FRE and max-product operator. Also, Ghodousian et al. [14,15,18] presented GA algorithms to solve the non-linear problem with FRE constraints defined by Lukasiewicz, Dubois –Prade and Sugeno-Weber operators.

Generally, there are three important difficulties related to FRE or FRI problems. Firstly, in order to completely determine FREs and FRIs, we must initially find all the minimal solutions, and the finding of all the minimal solutions is an NP-hard problem. Secondly, a feasible region formed as FRE or FRI [21] is often a non-convex set. Finally, FREs and FRIs as feasible regions lead to optimization problems with highly non-linear constraints.Due to the above mentioned difficulties, although the analytical methods are efficient to find exact optimal solutions, they may also involve high computational complexity for high-dimensional problems (especially, if the simplification processes cannot considerably reduce the problem).

In this paper, we use the genetic algorithm proposed in [14] for solving problem (1), which keeps the search inside of the feasible region without finding any minimal solution and checking the feasibility of new generated solutions.For this purpose, the paper consists of three main parts. Firstly, we describe some structural details of FREs defined by the Hamacher t-norm such as the theoretical properties of the solutions set, necessary and sufficient conditions for the feasibility of the problem,some simplification processes and the existence of an especial convex subset of the feasible region.Then, by utilizing the convex subset, the GA can easily generate a random feasible initial population. Finally, we provide some statistical and experimental results to evaluate the performance of our algorithm. Since the feasibility of problem(1) is essentially dependent on the t-norm (Hamacher t-norm) used in the definition of the constraints, a method is also presented to construct feasible test problems. More precisely, we construct a feasible problem by randomly generating a fuzzy matrix A and a fuzzy vector b according to some criteria resulted from the necessary and sufficient conditions.It is proved that the max-Hamacher fuzzy relational equations constructed by this method is not empty. Moreover, a comparison is made between the proposed GA and the genetic algorithms presented in [29] and [42].

The remainder of the paper is organized as follows. Section 2 takes a brief look at some basic results on the feasible solutions set of problem (1). In Section 3, the GA algorithm is briefly described. Finally, in Section 4 the experimental results are demonstrated and a conclusion is provided in Section 5.

II. BASIC PROPERTIES OF MAX-HAMACHER FRE

2.1. Characterization of feasible solutions set

This section describes the basic definitions and structural properties concerning problem (1) that are used throughout the paper. For the sake of simplicity, let $S_{T_H^\alpha}(a_i,b_i)$ denote the feasible solutions set of i 'th equation, that is, $S_{T_H^\alpha}(a_i,b_i)=\left\{x\in[0,1]^n:\ \max_{j=1}^{n}\left\{T_H^\alpha(a_{ij},x_j)\right\}=b_i\right\}$. Also, let $S_{T_H^\alpha}(A,b)$ denote the feasible solutions set of problem (1). Based on the foregoing notations, it is clear that $S_{T_H^\alpha}(A,b)=\bigcap_{i\in I}S_{T_H^\alpha}(a_i,b_i)$.

Definition 1. For each $i\in I$, we define $J_i=\left\{j\in J:\ a_{ij}\ge b_i\right\}$.

According to definition 1, we have the following lemmas, which are easily proved by the monotonicity and identity law of t-norms, definition 1 and the definition of Frank t-norm.

Lemma 1. Let $i\in I$. If $j\notin J_i$, then $T_H^\alpha(a_{ij},x_j)<b_i$, $\forall x_j\in[0,1]$.

Lemma 2. Let $i\in I$ and $j\in J_i$.

(a) If $x_j>\dfrac{[\alpha+(1-\alpha)a_{ij}]b_i}{a_{ij}-(1-\alpha)(1-a_{ij})b_i}$ and $b_i\ne 0$, then $T_H^\alpha(a_{ij},x_j)>b_i$.

(b) If $x_j=\dfrac{[\alpha+(1-\alpha)a_{ij}]b_i}{a_{ij}-(1-\alpha)(1-a_{ij})b_i}$ and $b_i\ne 0$, then $T_H^\alpha(a_{ij},x_j)=b_i$.

(c) If $x_j<\dfrac{[\alpha+(1-\alpha)a_{ij}]b_i}{a_{ij}-(1-\alpha)(1-a_{ij})b_i}$ and $b_i\ne 0$, then $T_H^\alpha(a_{ij},x_j)<b_i$.

(d) If $a_{ij}=b_i=0$, then $T_H^\alpha(a_{ij},x_j)=b_i$, $\forall x_j\in[0,1]$.

(e) If $a_{ij}>b_i=0$, then $T_H^\alpha(a_{ij},x_j)=b_i$ for $x_j=0$, and $T_H^\alpha(a_{ij},x_j)>b_i$ for $0<x_j\le 1$.

Lemma 3. For a fixed $i\in I$, $S_{T_H^\alpha}(a_i,b_i)\ne\varnothing$ if and only if $J_i\ne\varnothing$.

Proof. The proof is similar to the proof of Lemma 3 in [14].□

Definition 2. Suppose that $i\in I$ and $S_{T_H^\alpha}(a_i,b_i)\ne\varnothing$ (hence, $J_i\ne\varnothing$ from lemma 3). Let $\hat{x}_i=[(\hat{x}_i)_1,(\hat{x}_i)_2,\dots,(\hat{x}_i)_n]\in[0,1]^n$ where the components are defined as follows:

$$(\hat{x}_i)_k=\begin{cases}\dfrac{[\alpha+(1-\alpha)a_{ik}]b_i}{a_{ik}-(1-\alpha)(1-a_{ik})b_i} & k\in J_i,\ b_i\ne 0\\ 0 & k\in J_i,\ a_{ik}>b_i=0\\ 1 & otherwise\end{cases},\ \forall k\in J$$

Also, for each $j \in J_i$, we define $\breve{x}_i(j) = [\breve{x}_i(j)_1, \breve{x}_i(j)_2, ..., \breve{x}_i(j)_n] \in [0,1]^n$ such that

$$\breve{x}_i(j)_k = \begin{cases} \dfrac{[\alpha + (1-\alpha)a_{ik}]b_i}{a_{ik} - (1-\alpha)(1-a_{ik})b_i} & b_i \neq 0 \ and \ k = j \\ 0 & otherwise \end{cases}, \ \forall k \in J$$

Theorem 1. Let $i \in I$. If $S_{T_H^\alpha}(a_i, b_i) \neq \varnothing$, then $S_{T_H^\alpha}(a_i, b_i) = \bigcup_{j \in J_i} [\, \breve{x}_i(j), \hat{x}_i \,]$.

Proof. For a more general case, see Corollary 2.3 in [21].□

Definition 3. Let $\hat{x}_i$ ($i \in I$) be the maximum solution of $S_{T_H^\alpha}(a_i, b_i)$. We define $\overline{X} = \min_{i \in I}\{\hat{x}_i\}$.

Definition 4. Let $e: I \to J_i$ so that $e(i) = j \in J_i$, $\forall i \in I$, and let E be the set of all vectors e. For the sake of convenience, we represent each $e \in E$ as an m–dimensional vector $e = [j_1, j_2, ..., j_m]$ in which $j_k = e(k)$.

Definition 5. Let $e = [j_1, j_2, ..., j_m] \in E$. We define $\underline{X}(e) = [\underline{X}(e)_1, \underline{X}(e)_2, ..., \underline{X}(e)_n] \in [0,1]^n$, where $\underline{X}(e)_j = \max_{i \in I}\{\breve{x}_i(e(i))_j\} = \max_{i \in I}\{\breve{x}_i(j_i)_j\}$, $\forall j \in J$.

From the relation $S_{T_F^s}(A,b) = \bigcap_{i \in I} S_{T_F^s}(a_i, b_i)$ and Theorem 1, the following theorem is easily attained.

Theorem 2. $S_{T_H^\alpha}(A,b) = \bigcup_{e \in E} [\, \underline{X}(e), \overline{X} \,]$.

As a consequence, it turns out that $\overline{X}$ is the unique maximum solution and $\underline{X}(e)$'s ($e \in E$) are the minimal solutions of $S_{T_H^\alpha}(A,b)$. Moreover, we have the following corollary that is directly resulted from theorem 2.

Corollary 1(first necessary and sufficient condition). $S_{T_H^\alpha}(A,b) \neq \varnothing$ if and only if $\overline{X} \in S_{T_H^\alpha}(A,b)$.

Example 1. Consider the problem below with Hamacher t-norm

$$\begin{bmatrix} 0.9 & 0.4 & 0.6 & 0.7 & 0.4 & 0.4 \\ 0.5 & 0.1 & 0.2 & 0.3 & 0.5 & 0.2 \\ 0.2 & 0.8 & 0.4 & 0.4 & 0.6 & 0.2 \\ 0.9 & 0.7 & 0.3 & 0.8 & 0.8 & 0.5 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \end{bmatrix} \varphi x = \begin{bmatrix} 0.7 \\ 0.5 \\ 0.6 \\ 0.8 \\ 0 \end{bmatrix}$$

where $\varphi(x,y) = T_H^2(x,y) = \dfrac{xy}{2 - x - y + xy}$ (i.e., $\alpha = 2$).By definition 1, we have $J_1 = \{1,4\}$, $J_2 = \{1,5\}$, $J_3 = \{2,5\}$, $J_4 = \{1,4,5\}$ and $J_5 = \{1,2,3,4,5,6\}$. The unique maximum solution and the minimal solutions of each equation are obtained by definition 2 as follows:

$\hat{x}_1 = [0.7938\,,1\,,1\,,1\,,1\,,1]$, $\hat{x}_2 = [1\,,1\,,1\,,1\,,1\,,1]$, $\hat{x}_3 = [1\,,0.7826\,,1\,,1\,,1\,,1]$,

$\hat{x}_4 = [0.8980\,,1\,,1\,,1\,,1\,,1]$, $\hat{x}_5 = [1\,,1\,,1\,,0\,,1\,,1]$.

$\breve{x}_1(1) = [0.7938\,,0\,,0\,,0\,,0\,,0]$, $\breve{x}_1(4) = [0\,,0\,,0\,,1\,,0\,,0]$,

$\breve{x}_2(1) = [1\,,0\,,0\,,0\,,0\,,0]$, $\breve{x}_2(5) = [0\,,0\,,0\,,0\,,1\,,0]$

$\breve{x}_3(2) = [0\,,0.7826\,,0\,,0\,,0\,,0]$, $\breve{x}_3(5) = [0\,,0\,,0\,,0\,,1\,,0]$

$\breve{x}_4(1)=[0.8980\,,0\,,0\,,0\,,0\,,0]$, $\breve{x}_4(4)=[0\,,0\,,0\,,1\,,0\,,0]$, $\breve{x}_4(5)=[0\,,0\,,0\,,0\,,1\,,0]$

$\breve{x}_5(j)=[0\,,0\,,0\,,0\,,0\,,0]$, $j\in\{1,2,3,4,5,6\}$

Therefore, by theorem 1 we have $S_{T_H^\alpha}(a_1,b_1)=[\breve{x}_1(1),\hat{x}_1]\bigcup[\breve{x}_1(4),\hat{x}_1]$, $S_{T_H^\alpha}(a_2,b_2)=[\breve{x}_2(1),\hat{x}_2]\bigcup[\breve{x}_2(5),\hat{x}_2]$, $S_{T_H^\alpha}(a_3,b_3)=[\breve{x}_3(2),\hat{x}_3]\bigcup[\breve{x}_3(5),\hat{x}_3]$ and $S_{T_H^\alpha}(a_4,b_4)=[\breve{x}_4(1),\hat{x}_4]\bigcup[\breve{x}_4(4),\hat{x}_4]\bigcup[\breve{x}_4(5),\hat{x}_4]$ and $S_{T_H^\alpha}(a_5,b_5)=[\mathbf{0}_{1\times 6},\hat{x}_5]$ where $\mathbf{0}_{1\times 6}$ is a zero vector.From definition 3, $\overline{X}=[0.79381\,,0.78261\,,1\,,0\,,1\,,1]$. It is easy to verify that $\overline{X}\in S_{T_H^\alpha}(A,b)$. Therefore, the above problem is feasible by corollary 1. Finally, the cardinality of set E is equal to 24 (definition 4). So, we have 24 solutions $\underline{X}(e)$ associated to 24 vectors e. For example, for $e=[1,5,5,5,2]$, we obtain $\underline{X}(e)=\max\{\breve{x}_1(1),\breve{x}_2(5),\breve{x}_3(5),\breve{x}_4(5),\breve{x}_5(2)\}$ from definition 5 that means $\underline{X}(e)=[0.79381\,,0\,,0\,,0\,,1\,,0]$.

2.2. Simplification processes

In practice, there are often some components of matrix A that have no effect on the solutions to problem (1). Therefore, we can simplify the problem by changing the values of these components to zeros. For this reason, various simplification processes have been proposed by researchers. We refer the interesting reader to [21] where a brief review of such these processes is given. Here, we present two simplification techniques based on the Hamacher t-norm.

Definition 6. If a value changing in an element, say a_{ij}, of a given fuzzy relation matrix A has no effect on the solutions of problem (1), this value changing is said to be an equivalence operation.

Corollary 2. Suppose that $T_H^\alpha(a_{ij_0},x_{j_0})<b_i$, $\forall x\in S_{T_H^\alpha}(A,b)$. In this case, it is obvious that $\max_{j=1}^{n}\{T_H^\alpha(a_{ij},x_j)\}=b_i$ is equivalent to $\max_{\substack{j=1\\ j\neq j_0}}^{n}\{T_H^\alpha(a_{ij},x_j)\}=b_i$,that is, "resetting a_{ij_0} to zero" has no effect on the solutions of problem (1) (since component a_{ij_0} only appears in the i 'th constraint of problem (1)). Therefore, if $T_H^\alpha(a_{ij_0},x_{j_0})<b_i$, $\forall x\in S_{T_H^\alpha}(A,b)$, then "resetting a_{ij_0} to zero" is an equivalence operation.

Lemma 4 (first simplification). Suppose that $j_0\notin J_i$, for some $i\in I$ and $j_0\in J$. Then, "resetting a_{ij_0} to zero" is an equivalence operation.

Proof. From corollary 2, it is sufficient to show that $T_H^\alpha(a_{ij_0},x_{j_0})<b_i$, $\forall x\in S_{T_H^\alpha}(A,b)$. But, from lemma 1 we have $T_H^\alpha(a_{ij_0},x_{j_0})<b_i$, $\forall x_{j_0}\in[0,1]$. Thus, $T_H^\alpha(a_{ij_0},x_{j_0})<b_i$, $\forall x\in S_{T_H^\alpha}(A,b)$. □

Lemma 5 (second simplification). Suppose that $j_0\in J_{i_1}$ and $b_{i_1}\neq 0$, where $i_1\in I$ and $j_0\in J$. If at least one of the following conditions hold, then "resetting $a_{i_1 j_0}$ to zero" is an equivalence operation:

(a) There exists some $i_2\in I$ ($i_1\neq i_2$) such that $j_0\in J_{i_2}$, $b_{i_2}\neq 0$ and

$$\frac{[\alpha+(1-\alpha)a_{i_2 j_0}]b_{i_2}}{a_{i_2 j_0}-(1-\alpha)(1-a_{i_2 j_0})b_{i_2}}<\frac{[\alpha+(1-\alpha)a_{i_1 j_0}]b_{i_1}}{a_{i_1 j_0}-(1-\alpha)(1-a_{i_1 j_0})b_{i_1}}.$$

(b) There exists some $i_2 \in I$ ($i_1 \neq i_2$) such that $b_{i_2} = 0$ and $a_{i_2 j_0} > 0$.

Proof. (a)Similar to the proof of lemma 4, we show that $T_H^\alpha(a_{i_1 j_0}, x_{j_0}) < b_i, \forall x \in S_{T_H^\alpha}(A,b)$. Consider an arbitrary feasible solution $x \in S_{T_H^\alpha}(A,b)$. Since $x \in S_{T_H^\alpha}(A,b)$, it turns out that $T_H^\alpha(a_{i_1 j_0}, x_{j_0}) > b_{i_1}$ never holds. So, assume that $T_H^\alpha(a_{i_1 j_0}, x_{j_0}) = b_{i_1}$. Since $b_{i_1} \neq 0$, from lemma 2 we conclude that $x_{j_0} = \dfrac{[\alpha + (1-\alpha)a_{i_1 j_0}]b_{i_1}}{a_{i_1 j_0} - (1-\alpha)(1-a_{i_1 j_0})b_{i_1}}$.So, by the assumption, we have $\dfrac{[\alpha + (1-\alpha)a_{i_2 j_0}]b_{i_2}}{a_{i_2 j_0} - (1-\alpha)(1-a_{i_2 j_0})b_{i_2}} < x_{j_0}$. Therefore, lemma 2 (part (a)) implies $T_H^\alpha(a_{i_2 j_0}, x_{j_0}) > b_{i_2}$ that contradicts $x \in S_{T_H^\alpha}(A,b)$.

(b) By the assumption, we have $j_0 \in J_{i_2}$. Now, the result similarly follows by a simpler argument. □

We give an example to illustrate the above two simplification processes.

Example 2. Consider the problem presented in example 1.From the first simplification (lemma 4), "resetting the following components a_{ij} to zeros" are equivalence operations: $a_{12}, a_{13}, a_{15}, a_{16}; a_{22}, a_{23}, a_{24}, a_{26}; a_{31}, a_{33}, a_{34}, a_{36}; a_{42}, a_{43}, a_{46}$; in all of these cases, $a_{ij} < b_i$, that is, $j \notin J_i$. Also, from the second simplification (lemma 5, part (a)), we can change the values of components a_{21} and a_{41} to zeros.For example, $a_{41} > b_1$ (i.e., $1 \in J_4$), $b_4 \neq 0$, $a_{11} > b_1$ (i.e., $1 \in J_1$), $b_1 \neq 0$ and

$$0.7938 = \frac{[\alpha + (1-\alpha)a_{11}]b_1}{a_{11} - (1-\alpha)(1-a_{11})b_1} < \frac{[\alpha + (1-\alpha)a_{41}]b_4}{a_{41} - (1-\alpha)(1-a_{41})b_4} = 0.8980$$

Moreover, from lemma 5 (part (b)), we can also change the values of components a_{14} and a_{44} to zeros with no effect on the solutions set of the problem (since $4 \in J_1 \cap J_4$, $b_i \neq 0$ ($i = 1,4$), and $b_5 = 0$ and $a_{54} > 0$).

In addition to simplifying the problem, a necessary and sufficient condition is also derived from lemma 5. Before formally presenting the condition, some useful notations are introduced. Let $\tilde{A}$ denote the simplified matrix resulted from A after applying the simplification processes (lemmas 4 and 5). Also, similar to definition 1, assume that $\tilde{J}_i = \{ j \in J : \tilde{a}_{ij} \geq b_i \}$ ($i \in I$) where $\tilde{a}_{ij}$ denotes (i, j)'th component of matrix $\tilde{A}$. The following theorem gives a necessary and sufficient condition for the feasibility of problem (1).

Theorem 3 (second necessary and sufficient condition). $S_{T_H^\alpha}(A,b) \neq \varnothing$ if and only if $\tilde{J}_i \neq \varnothing$, $\forall i \in I$.

Proof. Since $S_{T_H^\alpha}(A,b) = S_{T_H^\alpha}(\tilde{A},b)$ from lemmas 4 and 5, it is sufficient to show that $S_{T_H^\alpha}(\tilde{A},b) \neq \varnothing$ if and only if $\tilde{J}_i \neq \varnothing$, $\forall i \in I$. Let $S_{T_H^\alpha}(\tilde{A},b) \neq \varnothing$. Therefore, $S_{T_H^\alpha}(\tilde{a}_i, b_i) \neq \varnothing$, $\forall i \in I$, where $\tilde{a}_i$ denotes i'th row of matrix $\tilde{A}$. Now, lemma 3 implies $\tilde{J}_i \neq \varnothing$, $\forall i \in I$. Conversely, suppose that $\tilde{J}_i \neq \varnothing$, $\forall i \in I$. Again, by using lemma 3 we have $\tilde{J}_i \neq \varnothing$, $\forall i \in I$. By contradiction, suppose that $S_{T_H^\alpha}(\tilde{A},b) = \varnothing$. Therefore, $\overline{X} \notin S_{T_H^\alpha}(\tilde{A},b)$ from corollary 1, and then there exists $i_0 \in I$ such that $\overline{X} \notin S_{T_H^\alpha}(\tilde{a}_{i_0}, b_{i_0})$. Since $\max_{j \notin \tilde{J}_i} \{ T_H^\alpha(\tilde{a}_{i_0 j}, \overline{X}_j) \} < b_{i_0}$ (from lemma 1), we must have either $\max_{j \in \tilde{J}_i} \{ T_H^\alpha(\tilde{a}_{i_0 j}, \overline{X}_j) \} > b_{i_0}$ or $\max_{j \in \tilde{J}_i} \{ T_H^\alpha(\tilde{a}_{i_0 j}, \overline{X}_j) \} < b_{i_0}$. Anyway, since $\overline{X} \leq \hat{x}_{i_0}$ (i.e., $\overline{X}_j \leq (\hat{x}_{i_0})_j$,

$\forall j \in J$), we have $\max_{j\in \tilde{J}_{i_0}} \left\{T_H^\alpha(\tilde{a}_{i_0 j}, \overline{X}_j)\right\} \le \max_{j\in \tilde{J}_{i_0}} \left\{T_H^\alpha(\tilde{a}_{i_0 j}, (\hat{x}_{i_0})_j)\right\} = b_{i_0}$, and then the former case (i.e., $\max_{j\in \tilde{J}_i} \left\{T_H^\alpha(\tilde{a}_{i_0 j}, \overline{X}_j)\right\} > b_{i_0}$) never holds. Therefore, $\max_{j\in \tilde{J}_i} \left\{T_H^\alpha(\tilde{a}_{i_0 j}, \overline{X}_j)\right\} < b_{i_0}$ that implies $b_{i_0} \ne 0$ and $T_H^\alpha(\tilde{a}_{i_0 j}, \overline{X}_j) < b_{i_0}$, $\forall j \in \tilde{J}_{i_0}$. Hence, by lemma 2, we must have $\overline{X}_j < \frac{[\alpha+(1-\alpha)\tilde{a}_{i_0 j}]b_{i_0}}{\tilde{a}_{i_0 j} - (1-\alpha)(1-\tilde{a}_{i_0 j})b_{i_0}}$, $\forall j \in \tilde{J}_{i_0}$. On the other hand, $\frac{[\alpha+(1-\alpha)\tilde{a}_{i_0 j}]b_{i_0}}{\tilde{a}_{i_0 j} - (1-\alpha)(1-\tilde{a}_{i_0 j})b_{i_0}} \le 1$, $\forall j \in \tilde{J}_{i_0}$. Therefore, $\overline{X}_j < 1$, $\forall j \in \tilde{J}_{i_0}$, and then from definitions 2 and 3, for each $j \in \tilde{J}_{i_0}$ there must exists $i_j \in I$ such that either $j \in \tilde{J}_{i_j}$ and $\overline{X}_j = (\hat{x}_{i_j})_j = \frac{[\alpha+(1-\alpha)\tilde{a}_{i_j j}]b_{i_j}}{\tilde{a}_{i_j j} - (1-\alpha)(1-\tilde{a}_{i_j j})b_{i_j}}$ or $j \in \tilde{J}_{i_j}$ and $\tilde{a}_{i_j j} > b_{i_j} = 0$. Until now, we proved that $b_{i_0} \ne 0$ and for each $j \in \tilde{J}_{i_0}$, there exist $i_j \in I$ such that either $j \in \tilde{J}_{i_j}$ and $\frac{[\alpha+(1-\alpha)\tilde{a}_{i_j j}]b_{i_j}}{\tilde{a}_{i_j j} - (1-\alpha)(1-\tilde{a}_{i_j j})b_{i_j}} < \frac{[\alpha+(1-\alpha)\tilde{a}_{i_0 j}]b_{i_0}}{\tilde{a}_{i_0 j} - (1-\alpha)(1-\tilde{a}_{i_0 j})b_{i_0}}$ (because, $\frac{[\alpha+(1-\alpha)\tilde{a}_{i_j j}]b_{i_j}}{\tilde{a}_{i_j j} - (1-\alpha)(1-\tilde{a}_{i_j j})b_{i_j}} = \overline{X}_j < \frac{[\alpha+(1-\alpha)\tilde{a}_{i_0 j}]b_{i_0}}{\tilde{a}_{i_0 j} - (1-\alpha)(1-\tilde{a}_{i_0 j})b_{i_0}}$) or $b_{i_j} = 0$ and $\tilde{a}_{i_j j} > 0$. But in both cases, we must have $\tilde{a}_{i_0 j} = 0$ ($\forall j \in \tilde{J}_{i_0}$) from the parts (a) and (b) of lemma 5, respectively. Therefore, $\tilde{a}_{i_0 j} < b_{i_0} \ne 0$ ($\forall j \in \tilde{J}_{i_0}$) that is a contradiction. □

Remark 1. Since $S_{T_H^\alpha}(A,b) = S_{T_H^\alpha}(\tilde{A},b)$ (from lemmas 4 and 5), we can rewrite all the previous definitions and results in a simpler manner by replacing $\tilde{J}_i$ with J_i ($i \in I$).

III. THE PROPOSED GA

In this section, the genetic algorithm proposed in [14] is briefly discussed. Since the feasible region of problem (1) is non-convex, a convex subset of the feasible region is firstly introduced. Consequently, the proposed GA can easily generate the initial population by randomly choosing individuals from this convex feasible subset. At the last part of this section, a method is presented to generate random feasible max-Yager fuzzy relational equations.

3.1. Initialization

The initial population is given by randomly generating the individuals inside the feasible region.For this purpose, we firstly find a convex subset of the feasible solutions set, that is, we find set F such that $F \subseteq S_{T_H^\alpha}(A,b)$ and F is convex. Then, the initial population is generated by randomly selecting individuals from set F .

Definition 7. Suppose that $S_{T_H^\alpha}(\tilde{A},b) \ne \varnothing$. For each $i \in I$, let $\breve{x}_i = [(\breve{x}_i)_1, (\breve{x}_i)_2, ..., (\breve{x}_i)_n] \in [0,1]^n$ where the components are defined as follows:

$$(\breve{x}_i)_k = \begin{cases} \frac{[\alpha+(1-\alpha)a_{ik}]b_i}{a_{ik} - (1-\alpha)(1-a_{ik})b_i} & b_i \ne 0 \ and \ k \in \tilde{J}_i \\ 0 & otherwise \end{cases}, \ \forall k \in J$$

Also, we define $\underline{X} = \max_{i \in I} \{ \breve{x}_i \}$.

Example 3. Consider the problem presented in example 1, where $\overline{X} = [0.7938 , 0.7826 , 1 , 0 , 1 , 1]$. Also, according to example 2, the simplified matrix $\tilde{A}$ is

$$\tilde{A} = \begin{bmatrix} 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0.8 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \end{bmatrix}$$

From definition 7, we have

$\breve{x}_1 = [0.7938 , 0 , 0 , 0 , 0 , 0]$, $\breve{x}_2 = [0 , 0 , 0 , 0 , 1 , 0]$, $\breve{x}_3 = [0 , 0.7826 , 0 , 0 , 1 , 0]$,
$\breve{x}_4 = [0 , 0 , 0 , 0 , 1 , 0]$
, $\breve{x}_5 = [0 , 0 , 0 , 0 , 0 , 0]$

, and then $\underline{X} = \max_{i=1}^{5} \{ \breve{x}_i \} = [0.7938 , 0.7826 , 0 , 0 , 1 , 0]$. Therefore, set $F = [\underline{X} , \overline{X}]$ is obtained as a collection of intervals:

$$F = [\underline{X} , \overline{X}] = [0.7938 , 0.7826 , [0,1] , 0 , 1 , [0,1]]$$

By generating random numbers in the corresponding intervals, we acquire one initial individual: $x = [0.7938 , 0.7826 , 0.45 , 0 , 1 , 0.98]$.

According to lemma 6, the algorithm for generating the initial population is simply obtained as follows:

Algorithm 1 (Initial Population).

1. Get fuzzy matrix A, fuzzy vector b and population size S_{pop}.
2. If $\overline{X} \notin S_{T_H^\alpha}(A,b)$, then stop; the problem is infeasible (corollary1).
3. For $i = 1,2,...,S_{pop}$

   Generate a random $n-$dimensional solution $pop(i)$ in the interval $[\underline{X} , \overline{X}]$.

   End

3.2. Selection strategy

Suppose that the individuals in the population are sorted according to their ranks from the best to worst, that is, individual $pop(r)$ has rank r. The probability P_r of choosing the r 'th individual is given by the following formulas:

$$P_r = \frac{W_r}{\sum_{k=1}^{S_{pop}} W_k} \quad , \quad W_r = \frac{1}{\sqrt{2\pi}\, q\, S_{pop}} e^{-\frac{1}{2}\left(\frac{r-1}{q\, S_{pop}}\right)^2}$$

where the weight to be a value of the Gaussian function with argument r, mean 1, and standard deviation $q\, S_{pop}$, where q is a parameter of the algorithm.

3.3. Mutation operator

As usual, suppose that $S_{T_H^\alpha}(A,b) \neq \emptyset$. So, from theorem 3 we have $\tilde{J}_i \neq \emptyset$, $\forall i \in I$, where $\tilde{J}_i = \{ j \in J : \tilde{a}_{ij} \geq b_i \}$, $\forall i \in I$ (see definition1 and remark 1).

Definition 8. Let $I^{+}=\left\{ i\in I: b_i\neq 0 \right\}$. So, we define $D=\left\{ j\in J: \text{ if } \exists i\in I^{+} \text{ such that } j\in \tilde{J}_i \Rightarrow \left| \tilde{J}_i \right| >1 \right\}$, where $\left| \tilde{J}_i \right|$ denotes the cardinality of set $\tilde{J}_i$.

the mutation operator is defined as follows:

Algorithm 2 (Mutation operator).
1. Get the matrix $\tilde{A}$, vector b and a selected solution $\dot{x}=[\dot{x}_1,...,\dot{x}_n]$.
2. While $D\neq\varnothing$
   2.1. Set $x'\leftarrow x$.
   2.2. Randomly choose $j_0\in D$, and set $x'_{j_0}=0$.
   2.3. IF x' is feasible, go to Crossover operator; otherwise, set $D=D-\{j_0\}$.

3.4. Crossover operator
In section 2, it was proved that $\overline{X}$ is the unique maximum solution of $S_{T_H^{\alpha}}(A,b)$. By using this result, the crossover operator is stated as follows:

Algorithm 3 (Crossover operator).
1. Get the maximum solution $\overline{X}$, the new solution x' (generated by algorith 2) and one parent $pop(k)$ (for some $k=1,2,...,S_{pop}$).
2. Generate a random number $\lambda_1\in[0,1]$. Set $x_{new1}=\lambda_1 x'+(1-\lambda_1)\overline{X}$.
3. Let $\lambda_2=\min_{\substack{j=1\\ j\neq k}}^{S_{pop}} \| pop(k)-pop(j) \|$ and $d=\overline{X}-pop(k)$.
   Set $x_{new2}=pop(k)+\min\left\{\lambda_2,1\right\} d$.

3.5. Construction of test problems
There are usually several ways to generate a feasible FRE defined with different t-norms. In what follows, we present a procedure to generate random feasible max- Hamacher fuzzy relational equations:

Algorithm 4 (construction of feasible Max-Hamacher FRE).

1. Randomly select m columns $\{j_1, j_2, ..., j_m\}$ from $J = \{1,2,...,n\}$.
2. Generate vector b whose elements are random numbers from [0,1].
3. For $i \in \{1,2,...,m\}$

    Assign a random number from $[b_i, 1]$ to a_{ij_i}.

    End
4. For $i \in \{1,2,...,m\}$

    For each $k \in \{1,2,...,m\} - \{i\}$

    If $b_k = 0$

    Set $a_{k\,j_i} = 0$.

    Else

    Set $L = \dfrac{[\alpha + (1-\alpha)a_{ij_i}]b_i}{a_{ij_i} - (1-\alpha)(1-a_{ij_i})b_i}$.

    Assign a random number from $[0\ ,\ \dfrac{(L(1-\alpha)+\alpha)b_k}{L+(L-1)(1-\alpha)b_k}]$ to $a_{k\,j_i}$.

    End

    End

    End
5. For each $i \in \{1,2,...,m\}$ and each $j \notin \{j_1, j_2, ..., j_m\}$

    Assign a random number from [0,1] to a_{ij}.

    End

By the following theorem, it is proved that algorithm 4 always generates random feasible max-Hamacher fuzzy relational equations.

Theorem 4. The solutions set $S_{T_H^\alpha}(A,b)$ of FRE (with Hamacher t-norm) constructed by algorithm 4 is not empty.

Proof. According to step 3 of the algorithm, $j_i \in J_i$, $\forall i \in I$. Therefore, $J_i \neq \varnothing$, $\forall i \in I$. To complete the proof, we show that $j_i \in \tilde{J}_i$, $\forall i \in I$. By contradiction, suppose that the second simplification process reset a_{ij_i} to zero, for some $i \in I$. So, $b_i \neq 0$ and there must exists some $k \in I$ ($k \neq i$) such that either $j_i \in J_k$, $b_k \neq 0$ and $\dfrac{[\alpha + (1-\alpha)a_{k\,j_i}]b_k}{a_{k\,j_i} - (1-\alpha)(1-a_{k\,j_i})b_k} < \dfrac{[\alpha + (1-\alpha)a_{ij_i}]b_i}{a_{ij_i} - (1-\alpha)(1-a_{ij_i})b_i}$ or $b_k = 0$ and $a_{k\,j_i} > 0$. In the former case, we note that $a_{k\,j_i} > \dfrac{(L(1-\alpha)+\alpha)b_k}{L+(L-1)(1-\alpha)b_k}$, where $L = \dfrac{[\alpha + (1-\alpha)a_{ij_i}]b_i}{a_{ij_i} - (1-\alpha)(1-a_{ij_i})b_i}$. Anyway, both cases contradict step 4. □

IV. EXPERIMENTAL RESULTS AND COMPARISON WITH RELATED WORKS

In this section, we present the experimental results for evaluating the performance of our algorithm. Firstly, we apply our algorithm to 8 test problems described in Appendix A. The test problems have been randomly generated in different sizes by algorithm 4 given in section 3. Since the objective function is an ordinary nonlinear function, we take some objective functions from the well-known source: Test Examples for Nonlinear Programming Codes [30]. In section 5.2, we make a comparison against the related GAs proposed in [29] and [42]. To perform a fair comparison, we follow the same experimental setup for the parameters $\theta = 0.5$,

$\xi = 0.01$, $\lambda = 0.995$ and $\gamma = 1.005$ as suggested by the authors in [29] and [42]. Since the authors did not explicitly reported the size of the population, we consider $S_{pop} = 50$ for all the three GAs. As mentioned before, we set $q = 0.1$ in relation (2) for the current GA. Moreover, in order to compare our algorithm with max-min GA [42] (max-product GA [29]), we modified all the definitions used in the current GA based on the minimum t-norm (product t-norm). For example, we used the simplification process presented in [42] for minimum, and the simplification process given in [19,29] for product. Finally, 30 experiments are performed for all the GAs and for eight test problems reported in Appendix B, that is, each of the preceding GA is executed 30 times for each test problem. All the test problems included in Appendix A, have been defined by considering $\alpha = 2$ in T_H^α. Also, the maximum number of iterations is equal to 100for all the methods.

5.1. Performance of the max-Hamacher GA

To verify the solutionsfound by the max-Hamacher GA, the optimal solutionsof the test problems are also needed. Since $S_{T_H^\alpha}(A,b)$ is formed as the union of the finite number of convex closed cells (theorem 2), the optimal solutions are also acquired by the following procedure:

1. Computing all the convex cells of the Hamacher FRE.
2. Searching the optimal solution for each convex cell.
3. Finding the global optimum by comparing these local optimal solutions.

The computational results of the eight test problems (see Appendix A) are shown in Table 1 and Figures 1-8. In Table 1, the results are averaged over 30 runs and the average best-so-far solution, average mean fitness function and median of the best solution in the last iteration are reported.

Table 2 includes the best results found by the max-Hamacher GA and the above procedure.According to Table 2,the optimal solutions computed by the max-Hamacher GAand the optimal solutions obtained by the above procedure match very well. Tables 1 and 2, demonstrate the attractive ability of themax-Hamacher GAto detect the optimal solutions of problem (1). Also, the good convergence rate of the max-Hamacher GA could be concluded from Table 1 and figures 1-8.

Table 1. Results of applying the max-Hamacher GA to the eight test problems of Appendix A.
The results have been averaged over 30 runs. Maximum number of iterations=100.

Test problems	Average best-so-far	Median best-so-far	Average mean fitness
A.1	34.163872	34.163868	34.163887
A.2	-0.4090829	-0.4090829	-0.4089404
A.3	-1.0897527	-1.0897527	-1.0892696
A.4	6.154911	6.154911	6.155172
A.5	62.602713	62.599052	62.611324
A.6	-0.235645	-0.235645	-0.235643
A.7	-0.920109	-0.920109	-0.919835
A.8	99.210416	99.210354	99.215563

Table 2. Comparison of the solutions found by Max-Hamacher GA
and the optimal values of the test problems described in Appendix A.

Test problems	Solutions of max-Hamacher GA	Optimal values
A.1	34.163868	34.163868
A.2	-0.409082	-0.409086
A.3	-1.089752	-1.089756
A.4	6.154911	6.154906
A.5	62.599047	62.599047
A.6	-0.235645	-0.235648
A.7	-0.920109	-0.92010999.210352
A.8	99.210352	

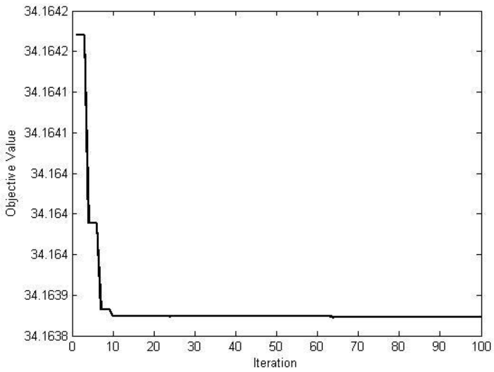


Figure 1. The performance of the max-Hamacher GA on test problem A.1.

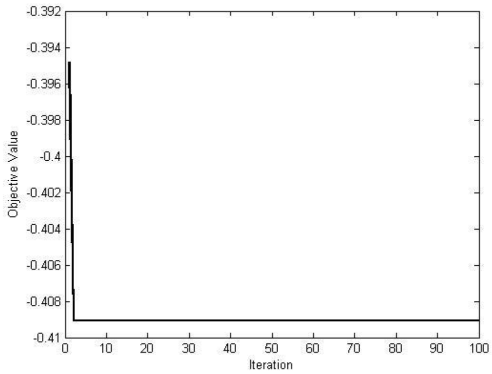


Figure 2. The performance of the max-Hamacher GA on test problem A.2.

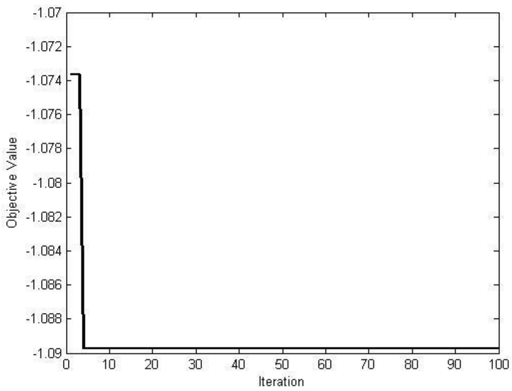


Figure 3. The performance of the max-Hamacher GA on test problem A.3.

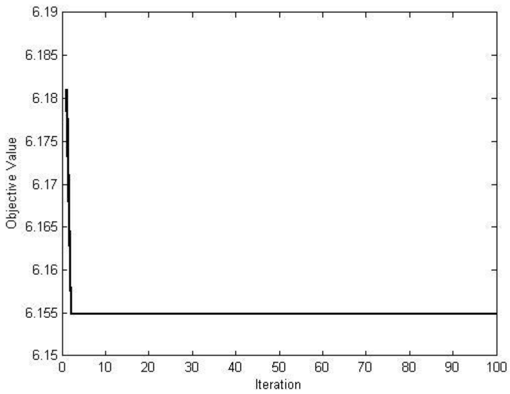


Figure 4. The performance of the max-Hamacher GA on test problem A.4.

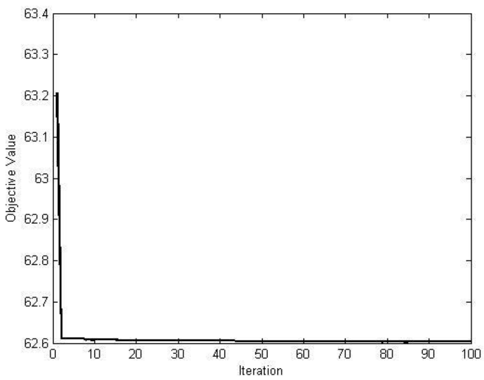


Figure 5. The performance of the max-Hamacher GA on test problem A.5.

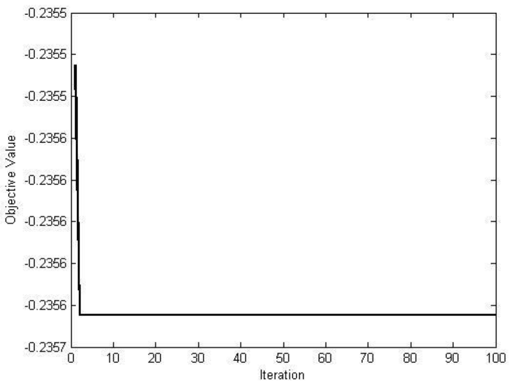
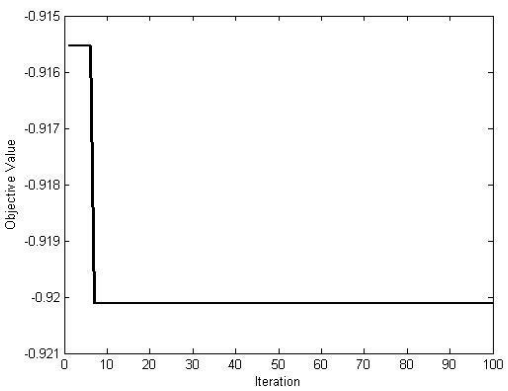


Figure 6. The performance of the max-Hamacher GA on test problem A.6.



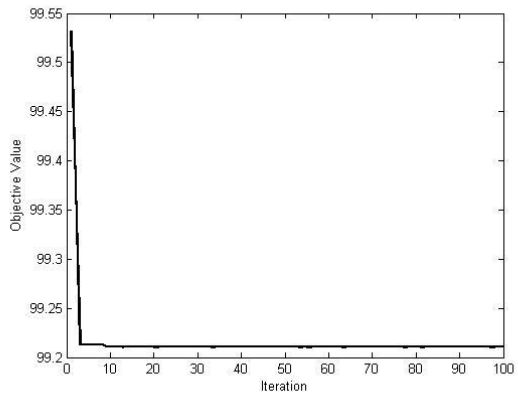


Figure 7. The performance of the max-Hamacher GA on test problem A.7.

Figure 8. The performance of the max-Hamacher GA on test problem A.8.

5.2. Comparisons with other works

As mentioned before, we can make a comparison between the current GA, max-min GA [42] and max-product GA [29]. For this purpose, all the test problems described in Appendix B have been designed in such a way that they are feasible for both the minimum and product t-norms.

The first comparison is against max-min GA, and we apply our algorithm (modified for the minimum t-norm) to the test problemsby considering φ as the minimum t-norm. The results are shown in Table 3 including the optimal objective values found by the current GA and max-min GA. As is shown in this table, the current GA finds better solutions for test problems 1, 5 and 6, and the same solutions for the other test problems.

Table 4 shows that the current GA finds the optimal values faster than max-min GA and hence has a higher convergence rate, even for the same solutions. The only exception is test problem 8 in which all the results are the same.In all the cases, results marked with "*" indicate the better cases.

The second comparison is against the max-productGA. In this case, we apply our algorithm (modified for the product t-norm) to the same test problems by considering φ as the product t-norm (Tables 5 and 6).

The results, in Tables 5 and 6, demonstrate that the current GA produces better solutions (or the same solutions with a higher convergence rate) when compared against max-productGAs for all the test problems.

Table 3. Best results found by our algorithm and max-min GA.

Test problems	Lu and Fang	Our algorithm
B.1	8.4296755	8.4296754*
B.2	-1.3888	-1.3888
B.3	0	0
B.4	5.0909	5.0909
B.5	71.1011	71.0968*
B.6	-0.3291	-0.4175*
B.7	-0.6737	-0.6737
B.8	93.9796	93.9796

Table 4. A Comparison between the resultsfound by the current GA and max-min GA.

Test problems		Max-min GA	Our GA
B.1	Average best-so-far Median best-so-far Average mean fitness	8.4297014 8.4296755 8.4308865	8.4296796* 8.4296755 8.4298745*
B.2	Average best-so-far Median best-so-far Average mean fitness	-1.3888 -1.3888 -1.3877	-1.3888 -1.3888 -1.3886*
B.3	Average best-so-far Median best-so-far Average mean fitness	0 0 7.1462e-07	0 0 0*
B.4	Average best-so-far Median best-so-far Average mean fitness	5.0909 5.0909 5.0910	5.0909 5.0909 5.0908*

B.5	Average best-so-far Median best-so-far Average mean fitness	71.1011 71.1011 71.1327	71.0969* 71.0968* 71.1216*
B.6	Average best-so-far Median best-so-far Average mean fitness	-0.3291 -0.3291 -0.3287	-0.4175* -0.4175* -0.4162*
B.7	Average best-so-far Median best-so-far Average mean fitness	-0.6737 -0.6737 -0.6736	-0.6737 -0.6737 -0.6737*
B.8	Average best-so-far Median best-so-far Average mean fitness	93.9796 93.9796 93.9796	93.9796 93.9796 93.9796

Table 5. Best results found by our algorithm and max-product GA.

Test problems	Hassanzadeh et al.	Our algorithm
B.1	13.61740269	13.61740246*
B.2	-1.5557	-1.5557
B.3	0	0
B.4	5.8816	5.8816
B.5	45.0650	45.0314*
B.6	-0.3671	-0.4622*
B.7	-2.470232	-2.470232
B.8	38.0195	38.0150*

Table 6. A Comparison between the results found by the current GA and max-product GA.

Test problems		Max-product GA	Our GA
B.1	Average best-so-far Median best-so-far Average mean fitness	13.61745044 13.61740371 13.61785924	13.61740502* 13.61740260* 13.61781613*
B.2	Average best-so-far Median best-so-far Average mean fitness	-1.5557 -1.5557 -1.5524	-1.5557 -1.5557 -1.5557*
B.3	Average best-so-far Median best-so-far Average mean fitness	0 0 1.5441e-05	0 0 0*
B.4	Average best-so-far Median best-so-far Average mean fitness	5.8816 5.8816 5.8823	5.8816 5.8816 5.8816*
B.5	Average best-so-far Median best-so-far Average mean fitness	45.0650 45.0650 45.1499	45.0315* 45.0314* 45.0460*
B.6	Average best-so-far Median best-so-far Average mean fitness	-0.3671 -0.3671 -0.3668	-0.4622* -0.4622* -0.4614*
B.7	Average best-so-far Median best-so-far Average mean fitness	-2.470232 -2.470232 -2.470175	-2.470232 -2.470232 -2.470213*
B.8	Average best-so-far Median best-so-far Average mean fitness	38.0195 38.0195 38.0292	38.0150* 38.0150* 38.0171*

V. CONCLUSION

In this paper, we investigated the resolution of FRE defined byHamacherfamily of t-norms and presented two necessary and sufficient conditions to determine the feasibility of the problem. Also, two simplification approaches (depending on the Hamacher t-norm) were proposed to simplify the problem. A nonlinear optimization problem was introduced in which the constraints were defined by the max-Hamacher fuzzy relational equations.A genetic algorithm was designed for solving the nonlinear optimization problems constrained by the max-Hamacher FRE. Moreover, we presented a method for generating feasible max-Hamacher FREs as test problems for the performance evaluation of the proposed algorithm. Experiments were performed with the proposed method in the generated feasible test problems. We conclude that the proposed GA can find the optimal solutions for all the cases with a great convergence rate. Moreover, a comparison was made between the proposed method and max-min and max-productGAs, which solve the nonlinear optimization problems subjected to the FREs defined by max-min and max-product compositions, respectively. The results showed that the proposed method (modified by minimum and product t-norms) finds better solutions compared with the solutions obtained by the other algorithms.

As future works, we aim at testing our algorithm in other type of nonlinear optimization problems whose constraints are defined as FRE or FRI with other well-known t-norms.

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Appendix A

Test Problem A.1:

$$f(x) = (x_1 + 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4$$

$$b^T = [0.7508\ ,\ 0.8667\ ,\ 0.1737]$$

$$A = \begin{bmatrix} 0.5466 & 0.4447 & 0.8618 & 0.9913 \\ 0.5714 & 1.0000 & 0.9840 & 0.9461 \\ 0.0739 & 0.4114 & 0.0604 & 0.1245 \end{bmatrix}$$

Test Problem A.2:

$$f(x) = x_1 - x_2 - x_3 - x_1x_3 + x_1x_4 + x_2x_3 - x_2x_4 + x_4x_5,$$

$$b^T = [0.3802\ ,\ 0.3713\ ,\ 0.9578\ ,\ 0.8987]$$

$$A = \begin{bmatrix} 0.0134 & 0.4195 & 0.4414 & 0.2261 & 0.8640 \\ 0.0595 & 0.8265 & 0.2358 & 0.2603 & 0.1849 \\ 0.9716 & 0.1958 & 0.9005 & 0.1110 & 0.4638 \\ 0.5184 & 0.6134 & 0.3578 & 0.9919 & 0.4770 \end{bmatrix}$$

Test Problem A.3:

$$f(x) = x_1x_2 - Ln(1 + x_3x_4x_5) - x_6,$$

$$b^T = [0.9064\ ,\ 0.4253\ ,\ 0.5050\ ,\ 0.6276]$$

$$A = \begin{bmatrix} 0.8257 & 0.7529 & 0.3805 & 0.4350 & 0.9634 & 0.4950 \\ 0.1376 & 0.8935 & 0.8517 & 0.9040 & 0.2221 & 0.1918 \\ 0.2914 & 0.6959 & 0.3200 & 0.2601 & 0.0035 & 0.6116 \\ 0.6592 & 0.5076 & 1.0009 & 0.3047 & 0.4999 & 0.4024 \end{bmatrix}$$

Test Problem A.4:

$$f(x) = x_1 + 2x_2 + 4x_5 + e^{x_1x_4 - x_6},$$

$b^T = [0.5849 , 0.4684 , 0.7908 , 0.3129 , 0.5777]$

$$A = \begin{bmatrix} 0.7531 & 0.6243 & 0.7585 & 0.8581 & 0.2904 & 0.6837 \\ 0.6443 & 0.1379 & 0.2702 & 0.8437 & 0.1479 & 0.3227 \\ 0.2535 & 0.8900 & 0.9201 & 0.3996 & 0.0430 & 0.2833 \\ 0.0669 & 0.2101 & 0.5484 & 0.2301 & 0.3199 & 0.0522 \\ 0.0563 & 0.2885 & 0.2611 & 0.8847 & 0.5889 & 0.3613 \end{bmatrix}$$

Test Problem A.5:

$$f(x) = \sum_{k=1}^{6} [100(x_{k+1} - x_k^2)^2 + (1 - x_k)^2],$$

$b^T = [0.2922 , 0.2526 , 0.3279 , 0.3855 , 0.8394]$

$$A = \begin{bmatrix} 0.4277 & 0.6059 & 0.1413 & 0.8197 & 0.3142 & 0.3271 & 0.1420 \\ 0.5121 & 0.2795 & 0.2053 & 0.3248 & 0.0851 & 0.2704 & 0.0956 \\ 0.1440 & 0.6632 & 0.9550 & 0.0402 & 0.3120 & 0.3677 & 0.0605 \\ 0.6556 & 0.7501 & 0.7828 & 0.1850 & 0.3841 & 0.3035 & 0.3935 \\ 0.7603 & 0.5252 & 1.1517 & 0.1699 & 0.1812 & 0.9584 & 0.3891 \end{bmatrix}$$

Test Problem A.6:

$$f(x) = -0.5(x_1x_4 - x_2x_3 + x_2x_6 - x_5x_6 + x_5x_4 - x_6x_7),$$

$b^T = [0.0387 , 0.0871 , 0.9195 , 0.4672 , 0.8911 , 0.5548]$

$$A = \begin{bmatrix} 0.0315 & 0.5460 & 0.6902 & 0.0179 & 0.1816 & 0.0356 & 0.0448 \\ 0.0079 & 0.8519 & 0.9319 & 0.0623 & 0.6220 & 0.1528 & 0.0114 \\ 0.9640 & 0.4568 & 0.2598 & 0.8875 & 1.1125 & 0.8720 & 0.1482 \\ 0.0063 & 0.1351 & 0.8162 & 0.2068 & 0.4893 & 0.7443 & 0.4999 \\ 0.0988 & 0.2774 & 1.2560 & 0.9306 & 1.4748 & 1.2097 & 0.7000 \\ 0.0414 & 0.4326 & 1.6203 & 0.5406 & 0.7725 & 0.5426 & 0.7002 \end{bmatrix}$$

Test Problem A.7:

$$f(x) = e^{x_1x_2x_3x_4x_5} - 0.5(x_1^3 + x_2^3 + x_6^3 + 1)^2 + 2x_7x_8,$$

$b^T = [0.1005 , 0.3808 , 0.1660 , 0.5817 , 0.9093 , 0.7985]$

$$A=\begin{bmatrix} 0.0310 & 0.6969 & 0.0185 & 0.0630 & 0.0919 & 0.0713 & 0.2496 & 0.4829 \\ 0.1073 & 0.6347 & 0.4281 & 0.2422 & 0.2589 & 0.6388 & 0.8061 & 0.5405 \\ 0.1249 & 0.2469 & 0.1687 & 0.1691 & 0.0796 & 0.9319 & 0.5389 & 0.4660 \\ 0.5950 & 0.3615 & 0.0740 & 0.5130 & 0.0819 & 0.4549 & 0.6721 & 0.9109 \\ 0.2556 & 1.4367 & 0.8898 & 0.9471 & 0.0732 & 0.5659 & 0.3055 & 0.5060 \\ 0.7614 & 1.1470 & 0.0670 & 0.5107 & 0.9179 & 0.2755 & 1.0347 & 0.8852 \end{bmatrix}$$

Test Problem A.8:

$$f(x)=(x_1-1)^2+(x_7-1)^2+10\sum_{k=1}^{7}(10-k)(x_k^2-x_{k+1})^2$$

$$b^T=[0.4237\ ,\ 0.3386\ ,\ 0.1688\ ,\ 0.6216\ ,\ 0.2097\ ,\ 0.2375\ ,\ 0.0649]$$

$$A=\begin{bmatrix} 0.2049 & 0.0773 & 0.2904 & 0.1781 & 0.2710 & 0.0364 & 0.6074 & 0.0767 \\ 0.2003 & 0.0072 & 0.0841 & 0.8016 & 0.4733 & 0.2531 & 0.2409 & 1.5668 \\ 0.0432 & 0.2902 & 0.0834 & 0.3294 & 0.3147 & 0.6609 & 0.1286 & 1.1558 \\ 0.6979 & 0.1751 & 0.3239 & 0.9019 & 0.0791 & 0.7741 & 0.6155 & 0.2101 \\ 0.0182 & 0.5373 & 0.0800 & 0.2726 & 0.1790 & 0.0183 & 0.1294 & 0.6273 \\ 0.1778 & 0.0600 & 0.2494 & 0.0040 & 0.2977 & 0.5861 & 0.2469 & 0.6972 \\ 0.0514 & 0.1384 & 0.0167 & 0.0765 & 0.0814 & 0.2829 & 0.0922 & 0.9899 \end{bmatrix}$$

Appendix B

Test Problem B.1:

$$f(x)=(x_1+10x_2)^2+5(x_3-x_4)^2+(x_2-2x_3)^4+10(x_1-x_4)^4$$

$$b^T=[0.2077\ ,\ 0.4709\ ,\ 0.8443]$$

$$A=\begin{bmatrix} 0.4302 & 0.4464 & 0.0741 & 0.0751 \\ 0.1848 & 0.1603 & 0.4628 & 0.5929 \\ 0.9049 & 0.1707 & 0.8746 & 0.4210 \end{bmatrix}$$

Test Problem B.2:

$$f(x)=x_1-x_2-x_3-x_1x_3+x_1x_4+x_2x_3-x_2x_4,$$

$$b^T=[0.4228\ ,\ 0.9427\ ,\ 0.9831]$$

$$A=\begin{bmatrix} 0.1280 & 0.7390 & 0.2852 & 0.2409 \\ 0.9991 & 0.7011 & 0.1688 & 0.9667 \\ 0.1711 & 0.6663 & 0.9882 & 0.6981 \end{bmatrix}$$

Test Problem B.3:

$$f(x)=x_1x_2x_3x_4x_5,$$

$$b^T=[0.6714\ ,\ 0.5201\ ,\ 0.1500]$$

$$A=\begin{bmatrix} 0.4424 & 0.3592 & 0.6834 & 0.6329 & 0.9150 \\ 0.6878 & 0.7363 & 0.7040 & 0.6869 & 0.2002 \\ 0.6482 & 0.3947 & 0.4423 & 0.0769 & 0.0175 \end{bmatrix}$$

Test Problem B.4:

$$f(x)=x_1+2x_2+4x_5+e^{x_1x_4},$$

$$b^T=[0.6855\ ,\ 0.5306\ ,\ 0.5975\ ,\ 0.2992]$$

$$A=\begin{bmatrix} 0.1025 & 0.7780 & 0.3175 & 0.9357 & 0.7425 \\ 0.0163 & 0.2634 & 0.5542 & 0.4579 & 0.9213 \\ 0.7325 & 0.2481 & 0.8753 & 0.2405 & 0.4193 \\ 0.1260 & 0.2187 & 0.6164 & 0.7639 & 0.2962 \end{bmatrix}$$

Test Problem B.5:

$$f(x)=\sum_{k=1}^{6}[100(x_{k+1}-x_k^2)^2+(1-x_k)^2],$$

$$b^T=[0.5846\ ,\ 0.8277\ ,\ 0.4425\ ,\ 0.8266]$$

$$A=\begin{bmatrix} 0.1187 & 0.4147 & 0.8051 & 0.3876 & 0.3643 & 0.7031 \\ 0.4761 & 0.8606 & 0.4514 & 0.0311 & 0.5323 & 0.1964 \\ 0.6618 & 0.2715 & 0.3826 & 0.0302 & 0.7117 & 0.1784 \\ 0.9081 & 0.1459 & 0.7896 & 0.9440 & 0.8715 & 0.1265 \end{bmatrix}$$

Test Problem B.6:

$$f(x)=-0.5(x_1x_4-x_2x_3+x_2x_6-x_5x_6+x_5x_4-x_6x_7),$$

$$b^T=[0.9879\ ,\ 0.6321\ ,\ 0.8082\ ,\ 0.6650]$$

$$A=\begin{bmatrix} 0.0832 & 0.3312 & 0.4580 & 0.7001 & 0.8287 & 0.9978 & 0.1876 \\ 0.3904 & 0.4277 & 0.2302 & 0.1373 & 0.4850 & 0.3495 & 0.8831 \\ 0.2393 & 0.8619 & 0.2734 & 0.8265 & 0.6598 & 0.4328 & 0.9315 \\ 0.4863 & 0.3787 & 0.6748 & 0.9301 & 0.4564 & 0.5893 & 0.8943 \end{bmatrix}$$

Test Problem B.7:

$$f(x)=e^{x_1x_2x_3x_4x_5}-0.5(x_1^3+x_2^3+x_6^3+1)^2,$$

$$b^T=[0.9521\ ,\ 0.0309\ ,\ 0.8627\ ,\ 0.8343\ ,\ 0.6290]$$

$$A = \begin{bmatrix} 0.9869 & 0.0805 & 0.8373 & 0.1417 & 0.9988 & 0.6320 \\ 0.0139 & 0.0169 & 0.0182 & 0.4379 & 0.0295 & 0.5095 \\ 0.2497 & 0.6914 & 0.8961 & 0.3504 & 0.8225 & 0.2433 \\ 0.9691 & 0.6170 & 0.5921 & 0.4785 & 0.5994 & 0.5714 \\ 0.6197 & 0.6298 & 0.2372 & 0.5874 & 0.2560 & 0.9817 \end{bmatrix}$$

Test Problem B.8:

$$f(x) = (x_1 - 1)^2 + (x_7 - 1)^2 + 10\sum_{k=1}^{6}(10-k)(x_k^2 - x_{k+1})^2$$

$$b^T = [0.7840\ ,\ 0.4648\ ,\ 0.8864\ ,\ 0.8352\ ,\ 0.9839]$$

$$A = \begin{bmatrix} 0.8522 & 0.2376 & 0.3586 & 0.7260 & 0.8891 & 0.2771 & 0.1316 \\ 0.4673 & 0.8176 & 0.1173 & 0.5350 & 0.1426 & 0.0020 & 0.2892 \\ 0.9707 & 0.4058 & 0.7248 & 0.1826 & 0.6193 & 0.8108 & 0.9630 \\ 0.8412 & 0.4663 & 0.7011 & 0.1124 & 0.6848 & 0.9434 & 0.4656 \\ 0.0785 & 0.9515 & 0.9997 & 0.0028 & 0.4982 & 0.6384 & 0.3852 \end{bmatrix}$$

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