A mixed finite element approximation of the Stokes equations with the boundary condition of type (D+N)

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ABSTRACT

In this paper we introduced the Stokes equations with a boundary condition of type (D+N).

The weak formulation obtained is a problem of saddle point type. We have shown the existence and uniqueness of the solution of this problem. We used the discretization by mixed finite element method with a posteriori error estimation of the computed solutions. In order to evaluate the performance of the method, the numerical results are compared with some previously published works or with others coming from commercial code like Adina system.

Keywords **- Stokes Equations, Mixed Finite element method, a posteriori error estimation, Adina system**.

I. INTRODUCTION

In modeling flow in porous media, it is essential to use a discretization method which satisfies the physics of the problem, i.e. conserve mass locally and preserve continuity of flux. The Raviart-Thomas Mixed Finite Element (MFE) method of lowest order satisfies these properties. Moreover, both the pressure and the velocity are approximated with the same order of convergence [4, 6]. The discretization of the velocity is based on the properties of Raviart-Thomas. Other works have been introduced by Brezzi, Fortin, Marini, Dougla and Robert [4, 5, 7].

This method was widely used for the prediction of the behavior of fluid in the hydrocarbons tank.

A posteriori error analysis in problems related to fluid dynamics is a subject that has received a lot of attention during the last decades. In the conforming case there are several ways to define error estimators by using the residual equation. in particular, for the Stokes problem, M. Ainsworth, J. Oden [9], R.E. Bank, B.D. Welfert [10], C. Crestensen, S.A. Funken [11], D. Kay, D. Silvester [12] and R. Verfurth [13], introduced several error estimators and provided that that they are equivalent to the energy norm of the errors. Other works for the stationary Navier-Stokes problem have been introduced in [14, 17, 18, 20, 16].

This paper describes a numerical solution of Stokes equations with a boundary condition noted (D+N). For the equations, we offer a choice of tow-dimensional domains on which the problem can be posed, along with boundary conditions and other aspects of the problem, and a choice of finite element discretization on a rectangular element mesh. The plan of the paper is as follows. The model problem is described in sections II, In Section III, we prove the existence and uniqueness of the solution of the weak formulation obtained, followed by a mixed finite element discretization for the Stokes equations in section IV. In section V we consider a posteriori error bounds of the computed solution, and numerical experiments are carried out in section VI.

II. GOVERNING EQUATIONS

We consider the Stokes equations for the flow;

$$-\nabla^2\vec{u} + \nabla p = \vec{f} \qquad in\ \Omega \tag{1}$$

$$\nabla.\vec{u} = 0 \qquad in\ \Omega \tag{2}$$

$$(D+N): \qquad b_0\vec{u} + \frac{\partial\vec{u}}{\partial n} - \vec{n}p = \vec{t} \quad in\ \Gamma \tag{3}$$

$\vec{u}$ is the fluid velocity, p is the pressure field. ∇ is the gradient, $\nabla.$ is the divergence and ∇^2 is the laplacien operator, $\vec{f} \in [L^2(\Omega)]^2$. Ω is a bounded and connected domain of $\mathbb{R}^2$ with a Lipchitz continuous boundary $\Gamma = \partial\Omega.$ where $\vec{n}$ denote the outward pointing normal to the boundary, and

$\vec{t} \in [L^2(\Gamma)]^2$. b_0 is a function defined and bounded on $\boldsymbol{\Gamma}$ verify:

$$\exists\, \alpha_0 > 0 \text{ such that } b_0 \geq \alpha_0 \text{ almost everywhere.} \tag{4}$$

We define the spaces:

$$h^1(\Omega) = \left\{u: \Omega \to \mathbb{R}/u\,;\ \frac{\partial u}{\partial x}\,;\ \frac{\partial u}{\partial y} \in L^2(\Omega)\right\} \tag{5}$$

$$H^1(\Omega) = [\,h^1(\Omega)]^2 \tag{6}$$

.

$$h^2(\Omega) = \left\{u: \Omega \to \mathbb{R}/u\,;\ \frac{\partial u}{\partial x_i}\,;\ \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega);\ \ i,j = 1,2\right\} \tag{7}$$

Then the standard weak formulation of the Stokes flow problem (1)-(2)-(3) is the following:

$$\begin{cases} find\ (\vec{u},p) \in H^1(\Omega)\times L^2(\Omega)\ such\ that: \\ \int_\Omega \nabla\vec{u}:\nabla\vec{v}\, d\Omega + \int_\Gamma b_0\vec{u}.\vec{v}\, d\gamma - \int_\Omega p\nabla.\vec{v}\, d\Omega = \int_\Gamma \vec{t}.\vec{v}\, d\gamma + \int_\Omega \vec{f}.\vec{v}\, d\Omega \text{ for all } \vec{v}\in H^1(\Omega) \\ -\int_\Omega q\nabla.\vec{u}\, d\Omega = 0 \quad \text{for all} \quad q\in L^2(\Omega) \end{cases} \tag{8}$$

Let the bilinear forms $a: H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ and $b: H^1(\Omega)\times L^2(\Omega)\to\mathbb{R}$

$$a(\vec{u},\vec{v}) = \int_\Omega \nabla\vec{u}:\nabla\vec{v}\, d\Omega + \int_\Gamma b_0\vec{u}.\vec{v}\, d\gamma \tag{9}$$

$$b(\vec{u},q) = -\int_\Omega q\nabla.\vec{u}\, d\Omega\,. \tag{10}$$

Given the functional $l: H^1(\Omega)\to\mathbb{R}$,

$$l(\vec{v}) = \int_\Gamma \vec{t}.\vec{v}\, d\gamma + \int_\Omega \vec{f}.\vec{v}\, d\Omega\,. \tag{11}$$

The underlying weak formulation (8) may be restated as

$$\begin{cases} find\ (\vec{u},p)\in H^1(\Omega)\times L^2(\Omega)\ such\ that: \\ a(\vec{u},\vec{v}) + b(\vec{u},q) = l(\vec{v}) \quad for\ all \in \vec{v}\ H^1(\Omega) \\ b(\vec{u},q) = 0 \quad for\ all\ q\in L^2(\Omega) \end{cases} \tag{12}$$

III. The existence and uniqueness of the solution

In this section we will show that the problem (12) has exactly one solution $(\vec{u},p)\in H^1(\Omega)\times L^2(\Omega)$.

It suffices to verify that the bilinear form a is positive, continuous and $H^1(\Omega)-$elliptic, and the bilinear form b is continuous and satisfies the inf-sup condition (see theorem 6.8 in [1]).

Theorem 3.1. $H^1(\Omega)$ is a real Hilbert space, with norm denoted by $\|.\|_{J,\Omega}$, for the scalar product:

$$\langle\vec{u},\vec{v}\rangle_J = \int_\Omega \nabla\vec{u}:\nabla\vec{v}\, d\Omega + \int_\Gamma \vec{u}.\vec{v}\, d\gamma \tag{13}$$

$$\|\vec{u}\|_{J,\Omega} = \langle\vec{u},\vec{u}\rangle_J^{\frac{1}{2}} = \left(\|\nabla\vec{u}\|_{0,\Omega}{}^2 + \|\vec{u}\|_{0,\Gamma}{}^2\right)^{\frac{1}{2}}. \tag{14}$$

To prove this theorem we need the following lemma.

Lemma 3.2. There are two strictly positive constants c_1 and c_2 such that:

$$c_1\|\vec{v}\|_{1,\Omega} \le \|\vec{v}\|_{J,\Omega} \le c_2\|\vec{v}\|_{1,\Omega}\ for\ all\ \vec{v}\in H^1(\Omega)\,. \tag{15}$$

Proof. The mapping $\gamma_0: H^1(\Omega)\to L^2(\Gamma)$ is continuous (see the theorem 1.2 in [21]), then there exists $c>0\ such\ that:$

$$\|\vec{v}\|_{0,\Gamma} \le c\|\vec{v}\|_{1,\Omega}\ for\ all\ \vec{v}\in H^1(\Omega)$$

We have also: $\|\nabla\vec{v}\|_{0,\Omega} \le \|\vec{v}\|_{1,\Omega}\ for\ all\ \vec{v}\in H^1(\Omega)$, then

$$\|\vec{v}\|_{J,\Omega} = \left(\|\nabla\vec{v}\|_{0,\Omega}{}^2 + \|\vec{v}\|_{0,\Gamma}{}^2\right)^{\frac{1}{2}} \le c_2\|\vec{v}\|_{1,\Omega}$$

$for\ all\ \vec{v}\in H^1(\Omega)$, with $c_2 = (c^2+1)^{\frac{1}{2}}$

On the other hand, there exists a constant $l>0\ such\ that$

$$\|\vec{v}\|_{0,\Omega} \le l(\|\nabla\vec{v}\|_{0,\Omega} + \|\vec{v}\|_{0,\Gamma})\ for\ all\ \vec{v}\in H^1(\Omega)$$

(see 5.55 in [1])

i.e.

$$\|\vec{v}\|_{0,\Omega}^2 \le l^2\left(\|\nabla\vec{v}\|_{0,\Omega} + \|\vec{v}\|_{0,\Gamma}\right)^2 \le 2l^2\left(\|\nabla\vec{v}\|_{0,\Omega}^2 + \|\vec{v}\|_{0,\Gamma}^2\right) = 2l^2\|\vec{v}\|_{J,\Omega}$$

We have also: $\|\nabla\vec{v}\|_{0,\Omega} \le \|\vec{v}\|_{1,\Omega}\ for\ all\ \vec{v}\in H^1(\Omega)$.

Then, $c_1\|\vec{v}\|_{1,\Omega} \le \|\vec{v}\|_{J,\Omega}$ for all $\vec{v}\in H^1(\Omega)$, with $c_1 = \left(\frac{1}{2l^2+1}\right)^{\frac{1}{2}}$.

Proof of theorem 3.1. Ω is a bounded and connected domain of $\mathbb{R}^2$, then it is easy to verify that $\langle.,.\rangle_J$ is a scalar product, i.e. $(H^1(\Omega), \|.\|_{J,\Omega})$ is an Euclidean space.

We have $(H^1(\Omega), \|.\|_{1,\Omega})$ is a real Hilbert, then it is complete. By the lemma 3.2, $\|.\|_{1,\Omega}$ and $\|.\|_{J,\Omega}$ are two equivalents norms, then $(H^1(\Omega), \|.\|_{J,\Omega})$ is complete, therefore it is a real Hilbert space.

Theorem 3.3. $i)$ a is continuous.

$ii)$ a is $H^1(\Omega)-$elliptic for the norm $\|.\|_{J,\Omega}$, i.e. there exists a constant $\alpha>0\ such\ that$

$$a(\vec{v},\vec{v}) \ge \alpha\|\vec{v}\|_{J,\Omega}^2\ for\ all\ \vec{v}\in H^1(\Omega)\,. \tag{16}$$

Proof. $i)$ a is a scalar product, by Cauchy-Schwarz inequality we have

$$a(\vec{u},\vec{v}) \leq \left(a(\vec{u},\vec{u}) \times a(\vec{v},\vec{v})\right)^{\frac{1}{2}} \leq M \,\|\vec{u}\|_{J,\Omega}.\|\vec{v}\|_{J,\Omega}$$
$$for\ all\ \vec{u},\ \vec{v} \in H^1(\Omega) \qquad (17)$$

with $M = Max(1;\ \sup_{x\in\Gamma} b_0(x))$.

ii) Let $\vec{v} \in H^1(\Omega)$, using (4) then gives:

$$a(\vec{v},\vec{v}) = \int_\Omega (\nabla\vec{v})^2 \, d\Omega + \int_\Gamma b_0(\vec{v})^2 \, d\gamma \geq \alpha\|\vec{v}\|^2_{J,\Omega},$$
With $\alpha = \min(1;\alpha_0)$.

Theorem 3.4. *i) b is continuous.*

ii) b is satisfies the inf-sup: There exists a constant $\beta > 0$ such that

$$\inf_{0\neq q\in L^2(\Omega)} \sup_{\vec{0}\neq\vec{v}\in H^1(\Omega)} \frac{\int_\Omega q\nabla.\vec{v}\,d\Omega}{\|q\|_{0,\Omega}\|\vec{v}\|_{J,\Omega}} \geq \beta. \qquad (18)$$

To prove this theorem we need the following lemmas:

Lemma 3.5.

$$\nabla.: \left(H^1(\Omega), \|.\|_{J,\Omega}\right) \to \left(L^2(\Omega), \|.\|_{0,\Omega}\right)$$
$$\vec{v} \longmapsto \nabla.\vec{v}$$

is continuous linear mapping and $R(\nabla.) = L^2(\Omega)$.

Proof. It is clear that $\nabla.$ is a linear mapping. Remains to chow that it is continuous and $R(\nabla.) = L^2(\Omega)$.

We have: $\|\nabla.\vec{v}\|_{0,\Omega} \leq \sqrt{2}\|\nabla\vec{v}\|_{0,\Omega} \leq \sqrt{2}\|\vec{v}\|_{J,\Omega}$ for all $\vec{v} \in H^1(\Omega)$, then $\nabla.$ is continuous

Let $q \in L^2(\Omega)$, then $\int_\Omega q\, d\Omega \in \mathbb{R}$.

We set $k = \int_\Omega q\, d\Omega$,

then $q - \frac{k}{|\Omega|} \in L^2_0(\Omega) = \{f \in L^2(\Omega); \int_\Omega f\, d\Omega = 0\}$,

with $|\Omega|$ is the area of Ω.

Since $L^2_0(\Omega)$ is the range space of the linear mapping $\nabla.$: $H^1_0(\Omega) \to L^2_0(\Omega)$ (see lemma 6.8 in [1]), then there exists $\vec{u} \in H^1_0(\Omega)$ such that $\nabla.\vec{u} = q$.

Let $\vec{u}_0 = \left(\frac{k}{|\Omega|}x; 0\right)$ for all $(x, y) \in \mathbb{R}^2$. we have $\nabla.(\vec{u} + \vec{u}_0) = q$, where $R(\nabla.) = L^2(\Omega)$.

Lemma 3.6. There exists a constant $\beta > 0$ such that: for all $q \in L^2(\Omega)$ there exists $\vec{v} \in H^1(\Omega)$ such that $\nabla.\vec{v} = q$ and $\beta\|\vec{v}\|_{J,\Omega} \leq \|q\|_{0,\Omega}$.

Proof. By the lemma 3.5, $R(\nabla.) = L^2(\Omega)$, then $R(\nabla.)$ is closed in $L^2(\Omega)$, therefore there exists $\beta > 0$ such that , for all $q \in L^2(\Omega)$ there exists $\vec{v} \in H^1(\Omega)$ such that $\nabla.\vec{v} = q$ and $\beta\|\vec{v}\|_{J,\Omega} \leq \|q\|_{0,\Omega}$ (see the lemma A.3 [1]).

Proof of theoreom 3.4. i) Let $(\vec{v}, q) \in H^1(\Omega) \times L^2(\Omega)$; we have

$$b(\vec{v},q) = -\int_\Omega q\nabla.\vec{v}\,d\Omega \leq \|q\|_{0,\Omega}\|\nabla.\vec{v}\|_{0,\Omega} \leq \sqrt{2}\|q\|_{0,\Omega}\|\nabla\vec{v}\|_{0,\Omega} \leq \sqrt{2}\|q\|_{0,\Omega}\|\vec{v}\|_{J,\Omega}$$

Then b is continuous.

ii) Let $q \in L^2(\Omega) - \{0\}$, by lemma 3.6, there exists $\vec{v'} \in H^1(\Omega)$ such that $\nabla.v' = q$ and $\beta\|\vec{v'}\|_{J,\Omega} \leq \|q\|_{0,\Omega}$. Then

$$\sup_{\vec{0}\neq\vec{v}\in H^1(\Omega)} \frac{\int_\Omega q\nabla.\vec{v}\,d\Omega}{\|\vec{v}\|_{J,\Omega}} \geq \frac{\int_\Omega q\nabla.\vec{v'}\,d\Omega}{\|\vec{v'}\|_{J,\Omega}} \geq \frac{\int_\Omega q^2\,d\Omega}{\|\vec{v'}\|_{J,\Omega}} = \frac{\|q\|^2_{0,\Omega}}{\|\vec{v'}\|_{J,\Omega}} \geq \beta.\|q\|_{0,\Omega}$$

We define the "big" symmetric bilinear form

$$B\left((\vec{u},p);(\vec{v},q)\right) = a(\vec{u},\vec{v}) + b(\vec{v},p) + b(\vec{u},q) \qquad (19)$$

And the corresponding function $F\left((\vec{v},q)\right) = l(\vec{v})$, choosing the successive test vectors $(\vec{v},0)$ and $(\vec{0},q)$ shows that the stokes problem (12) can be rewritten in the form:

$$find\ (\vec{u},p)\epsilon H^1(\Omega) \times L^2(\Omega)\ such\ that$$

$$B\left((\vec{u},p);(\vec{v},q)\right) = F\left((\vec{v},q)\right)\ for\ all\ (\vec{v},q)\epsilon H^1(\Omega)\times L^2(\Omega). \qquad (20)$$

The bilinear form is positive continuous and $H^1(\Omega) - elliptic$, and the bilinear form b is continuous and satisfies the inf-sup condition. Then the problem (12) is well-posed and the ''B-stability bound'' [1], given below:

Proposition 3.7. $for\ all\ (\vec{w},s)\epsilon H^1(\Omega) \times L^2(\Omega)$, we have that:

$$\sup_{(\vec{v},q)\epsilon H^1(\Omega)\times L^2(\Omega)} \frac{B\left((\vec{w},s);(\vec{v},q)\right)}{\|\vec{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \geq \gamma_D\left(\|\vec{w}\|_{J,\Omega} + \|s\|_{0,\Omega}\right) \qquad (21)$$

where γ_D depends only on the shape of the domain.

The bilinear form a is symmetric, and continuous and semi positive definite on $H^1(\Omega)$, in this case we say the problem (12) is a type of saddle-point problem.

The theorem (3.3) and (3.4) ensure the existence and uniqueness of the solution of the problem (12) (see the theorem 6. 2 in [1]). In the following section we will solve this problem by mixed finite element method.

IV. MIXED FINITE ELEMENT APPROXIMATION

A discrete weak formulation is defined using finite dimensional spaces $X^h_1 \subset H^1(\Omega)$ and $M^h \subset L_2(\Omega)$.

The discrete version of (12) is:

$$\begin{cases} find\ \overrightarrow{u_h}\ \epsilon X_1^h\ and\ p_h\ \epsilon M^h\ such\ that: \\ a(\overrightarrow{u_h},\overrightarrow{v_h}) + b(\overrightarrow{v_h},p_h) = l(\overrightarrow{v_h})\ \ \text{"for all"}\ \ \overrightarrow{v_h}\epsilon\ X_1^h \\ b(\overrightarrow{u_h},q_h) = 0 \qquad \text{"for all"}\ q_h\epsilon M^h \end{cases} \qquad (22)$$

We use a set of vector-valued basis functions $\{\overrightarrow{\varphi_j}\}_{j=1,\dots,n_u}$, so that

$$\overrightarrow{u_h} = \sum_{j=1}^{n_u} u_j\overrightarrow{\varphi_j} \qquad (23)$$

We introduce a set of pressure basis functions $\{\psi_k\}_{k=1,\dots,n_p}$ and set

$$p_h = \sum_{k=1}^{n_p} p_k\psi_k \qquad (24)$$

Where n_u and n_p are the numbers of velocity and pressure basis functions, respectively.

We find that the discrete formulation (22) can be expressed as a system of linear equations

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \qquad (25)$$

With

$$A = [a_{ij}];\ \ a_{ij} = \int_\Omega \nabla\overrightarrow{\varphi_i}:\nabla\overrightarrow{\varphi_j}d\Omega + \int_\Gamma b_0\overrightarrow{\varphi_i}.\overrightarrow{\varphi_j}d\gamma \qquad (26)$$

$$B = [b_{kj}];\ \ b_{kj} = -\int_\Omega \psi_k\nabla.\overrightarrow{\varphi_j}d\Omega \qquad (27)$$

for $i,j = 1,\dots,n_u$ and $k = 1,\dots,n_p$.

The right-hand side vectors in (4.4) are

$$f = [f_i];\ \ f_i = \int_\Omega \vec{f}\overrightarrow{\varphi_i}d\Omega + \int_\Gamma \overrightarrow{t_i}.\overrightarrow{\varphi_i}d\gamma \qquad (28)$$

for $i = 1,\dots,n_u$,

and the function pair$(\overline{u}_h,p_h)$ obtained by substituting the solution vectors $u \in \mathbb{R}^{n_u}$ and $p \in \mathbb{R}^{n_p}$ into (23) and (24) is the mixed finite element solution. The system (25)-(28) is henceforth referred to as the discrete stokes problem.

We use the iterative methods Minimum Residual Method (MINRES) for solving the symmetric system.

V. A RESIDUAL ERROR ESTIMATOR

In this section we assume that $\vec{f}$ and $\vec{t}$ are the polynomials.

Let $T_h\,; h > 0$, be a family of rectangulations of Ω. For any TϵT_h, ω_T is of rectangles sharing at least one edge with element T, $\widetilde{\omega}_T$ is the set of rectangles sharing at least one vertex with T. Also, for an element edge E, ω_E denotes the union of rectangles sharing E, while $\widetilde{\omega}_E$ is the set of rectangles sharing at least one vertex whit E.

Next, ∂T is the set of the four edges of T we denote by $\varepsilon(T)$ and N_T the set of its edges and vertices, respectively.

We let $\varepsilon_h = \cup_{T\epsilon T_h}\varepsilon(T)$ denotes the set of all edges split into interior and boundary edges.

$$\varepsilon_h = \varepsilon_{h,\Omega} \cup \varepsilon_{h,\Gamma}\,,$$

where

$$\varepsilon_{h,\Omega} = \{E\epsilon\varepsilon_h : E \subset \Omega\}\,,$$

$$\varepsilon_{h,\Gamma} = \{E\epsilon\varepsilon_h : E \subset \partial\Omega\}\,.$$

The bubble functions on the reference element $\tilde{T} = (0,1)\times(0,1)$ are defined as follows:

$$b_{\tilde{T}} = 2^4x(1-x)y(1-y)$$
$$b_{\tilde{E}_1,\tilde{T}} = 2^2x(1-x)(1-y)$$
$$b_{\tilde{E}_2,\tilde{T}} = 2^2y(1-y)x$$
$$b_{\tilde{E}_3,\tilde{T}} = 2^2y(1-x)$$
$$b_{\tilde{E}_4,\tilde{T}} = 2^2y(1-y)(1-x)$$

Here $b_{\tilde{T}}$ is the reference element bubble function, and $b_{\tilde{E}_i,\tilde{T}}$, $i = 1:4$ are reference edge bubble functions. For any TϵT_h, the element bubble functions is $b_T = b_{\tilde{T}}o\mathrm{F}_\mathrm{T}$ and the element edge bubble function is $b_{E_i,T} = b_{\tilde{E}_i,\tilde{T}}o\mathrm{F}_\mathrm{T}$ where F_T the affine map form $\tilde{T}\ to\ T$.

For an interior edge E$\epsilon\varepsilon_{h,\Omega}$, b_E is defined piecewise, so that $b_{E/T_i} = b_{E,T_i}$, $i = 1:2$, where $E = \overline{T}_1 \cap \overline{T}_2$. For a boundary edge $E\epsilon\ \varepsilon_{h,\Gamma}$, $b_E = b_{E,T}$, where T is the rectangle such that $E\epsilon\partial T$.

With these bubble functions, ceruse et al ([19], lemma 4.1] established the following lemma.

Lemma 5.1. Let T be an arbitrary rectangle in $T_h\ and\ E \in \partial T$.

For any$\vec{v}_T\epsilon P_{k_0}(T)\ and\ \vec{v}_E\epsilon P_{k_1}(E)$, the following inequalities hold.

$$c_k\|\vec{v}_T\|_{0,T} \le \left\|\vec{v}_Tb^{\frac{1}{2}}{}_T\right\|_{0,T} \le C_k\|\vec{v}_T\|_{0,T} \qquad (29)$$

$$|\vec{v}_Tb_T|_{1,T} \le C_kh_T^{-1}\|\vec{v}_T\|_{0,T} \qquad (30)$$

$$c_k\|\vec{v}_E\|_{0,E} \le \left\|\vec{v}_Eb^{\frac{1}{2}}{}_E\right\|_{0,E} \le C_k\|\vec{v}_E\|_{0,E} \qquad (31)$$

$$\|\vec{v}_Eb_E\|_{0,T} \le C_kh_E^{\frac{1}{2}}\|\vec{v}_E\|_{0,E} \qquad (32)$$

$$|\vec{v}_Eb_E|_{1,T} \le C_kh_E^{-\frac{1}{2}}\|\vec{v}_E\|_{0,E}\,, \qquad (33)$$

where c_k and C_k are tow constants which only depend on the element aspect ratio and the polynomial degrees k_0 and k_1.

Here, k_0 and k_1 are fixed and c_k and C_k can be associated with generic constants $c\ and\ C$. In addition, $\vec{v}_E$

which is only defined on the edge E also denotes its natural extension to the element T.

From the inequalities (32) and (33) we established the following lemma:

Lemma 5.2. Let T be an rectangle $and\ E \in \partial T \cap \varepsilon_{h,\Gamma}$. For any $\vec{v}_E \epsilon P_{k_1}(E)$, the following inequalities hold.

$$\|\vec{v}_E b_E\|_{J,T} \leq C h_E^{-\frac{1}{2}} \|\vec{v}_E\|_{0,E}. \quad (34)$$

Proof. Since $\vec{v}_E b_E = \vec{0}$ in the other three edges of rectangle T, it can be extended to the whole of Ω by setting $\vec{v}_E b_E = \vec{0}$ in $\Omega \backslash \bar{T}$, then

$$\|\vec{v}_E b_E\|_{1,T} = \|\vec{v}_E b_E\|_{1,\Omega}$$

$$and\ \|\vec{v}_E b_E\|_{J,T} = \|\vec{v}_E b_E\|_{J,\Omega}.$$

Using the inequalities (32), (33) and the lemma (3.2) gives

$$\|\vec{v}_E b_E\|_{J,T} = \|\vec{v}_E b_E\|_{J,\Omega} \leq c_2 \|\vec{v}_E b_E\|_{1,\Omega}$$

$$= c_2 \|\vec{v}_E b_E\|_{1,T}$$

$$= c_2 \left(\|\vec{v}_E b_E\|_{0,T}^2 + |\vec{v}_E b_E|_{1,T}^2 \right)^{\frac{1}{2}}$$

$$\|\vec{v}_E b_E\|_{J,T} \leq c_2 C_k (h_E + h_E^{-1})^{\frac{1}{2}} \|\vec{v}_E\|_{0,E}$$

$$\leq c_2 C_k (D^2+1)^{\frac{1}{2}} h_E^{-\frac{1}{2}} \|\vec{v}_E\|_{0,E}$$

$$\leq C h_E^{-\frac{1}{2}} \|\vec{v}_E\|_{0,E},,$$

where D is the diameter of Ω and $C = c_2 C_k (D^2+1)^{\frac{1}{2}}$.

We recall some quasi-interpolation estimates in the following lemma.

Lemma 5.3. Clement interpolation estimate: Given $\vec{v} \epsilon H^1(\Omega)$, let $\vec{v}_h \epsilon X_h$ be the quasi-interpolant of $\vec{v}$ defined by averaging as in [20].

For any TϵT_h, $\|\vec{v} - \vec{v}_h\|_{0,T} \leq C h_T |\vec{v}|_{1,\tilde{\omega}_T}$, (35)

and for all E$\epsilon \partial T$, $\|\vec{v} - \vec{v}_h\|_{0,E} \leq C h_E^{\frac{1}{2}} |\vec{v}|_{1,\tilde{\omega}_E}$. (36)

We let $(\vec{u}, p)$ denote the solution of (12) and let $(\vec{u}_h, p_h)$ denote the solution of (22) with an approximation on a rectangular subdivision T_h.

Our aim is to estimate the velocity and the pressure errors $\vec{e} = \vec{u} - \vec{u}_h; \varepsilon = p - p_h$.

The element contribution $\eta_{R,T}$ of the residual error estimator η_R is given by

.

$$\eta_{R,T}^2 = h_T^2 \|\vec{R}_T\|_{0,T}^2 + \|R_T\|_{0,T}^2 + \sum_{E \epsilon \partial T} h_E \|\vec{R}_E\|_{0,E}^2 \quad (37)$$

and the components in (37) are given by

$$\vec{R}_T = \{\vec{f} + \nabla^2 \vec{u}_h - \nabla p_h\}_{/T} \quad (38)$$

$$R_T = \{\nabla . \vec{u}_h\}_{/T} \quad (39)$$

$$\vec{R}_E = \begin{cases} \frac{1}{2} [\![\nabla \vec{u}_h - p_h I]\!] ; E \epsilon \varepsilon_{h,\Omega} \\ \vec{t} - \left(b_0 \vec{u}_h + \frac{\partial \vec{u}_h}{\partial \vec{n}_{E,T}} - p_h \vec{n}_{E,T} \right); E \epsilon \varepsilon_{h,\Gamma} \end{cases} \quad (40)$$

With the key contribution coming from the stress jump associated with an edge E adjoining elements T and S:

$$[\![\nabla \vec{u}_h - p_h I]\!] = \left((\nabla \vec{u}_h - p_h I)_{/T} - (\nabla \vec{u}_h - p_h I)_{/S} \right) \vec{n}_{E,T}$$

The global residual error estimator is given by:

$$\eta_R = \sqrt{\sum_{T \epsilon T_h} \eta_{R,T}^2}.$$

For any TϵT_h, and $E \in \partial T$, we define the following two functions:

$\vec{w}_T = \vec{R}_T b_T$; $\vec{w}_E = \vec{R}_E b_E$.

$Since\ \vec{w}_T = \vec{0}\ on\ \partial T, it$ can be extended to the whole of Ω by setting $\vec{w}_T = \vec{0}$ in Ω\T.

- $if\ E \epsilon \partial T \cap \varepsilon_{h,\Omega}\ then\ \vec{w}_E = \vec{0}\ in\ \partial \omega_E$.
- $if\ E \epsilon \partial T \cap \varepsilon_{h,\Gamma}\ then\ \vec{w}_E = \vec{0}$ in the other three edges of rectangle T.

With these two functions we have the following lemmas:

Lemma 5.4. for any TϵT_h we have:

$$\int_T (\nabla \vec{u} - pI) : \nabla \vec{w}_T d\Omega = \int_\Omega (\nabla \vec{u} - pI) : \nabla \vec{w}_T d\Omega = \int_T \vec{f} . \vec{w}_T d\Omega \quad (41)$$

Proof.

$$\int_T (\nabla \vec{u} - pI) : \nabla \vec{w}_T d\Omega = \int_{\partial T} (\nabla \vec{u} - pI) \vec{n} . \vec{w}_T d\gamma - \int_T (\nabla^2 \vec{u} - \nabla p) . \vec{w}_T d\Omega$$

Since $-\nabla^2 \vec{u} + \nabla p = \vec{f}$ in Ω and $\vec{w}_T = \vec{0}$ in Ω\T

then:

$$\int_T (\nabla \vec{u} - pI) : \nabla \vec{w}_T d\Omega = \int_\Omega (\nabla \vec{u} - pI) : \nabla \vec{w}_T d\Omega = \int_T \vec{f} . \vec{w}_T d\Omega.$$

Lemma 5.5. $i)\ if\ E \epsilon \partial T \cap \varepsilon_{h,\Omega}, we\ have:$

$$\int_{\omega_E} (\nabla \vec{u} - pI) : \nabla \vec{w}_E\, d\Omega = \int_{\omega_E} \vec{f} . \vec{w}_E\, d\Omega. \quad (42)$$

$ii) if\ E \epsilon \partial T \cap \varepsilon_{h,\Gamma}, we\ have:$

$$\int_T (\nabla \vec{u} - pI) : \nabla \vec{w}_E\, d\Omega = \int_{\partial T} \left(\vec{t} - b_0 \vec{u} \right) \vec{w}_E\, d\gamma + \int_T \vec{f} . \vec{w}_E\, d\Omega \quad (43)$$

Proof. *i)* The same proof of (41).

ii) if $E \epsilon \partial T \cap \varepsilon_{h,\Gamma}$, we have:

$$\int_T (\nabla\vec{u} - pI):\nabla \vec{w}_E \, d\Omega = \int_{\partial T} (\nabla\vec{u} - pI)\vec{n}_T.\vec{w}_E \, d\gamma - \int_T (\nabla^2\vec{u} - \nabla p).\vec{w}_E d\Omega$$

$$= \int_E (\nabla\vec{u} - pI)\vec{n}_T.\vec{w}_E \, d\gamma - \int_T (\nabla^2\vec{u} - \nabla p).\vec{w}_E d\Omega$$

Since $-\nabla^2\vec{u} + \nabla p = \vec{f}$ on Ω, and

$b_0\vec{u} + \frac{\partial \vec{u}}{\partial n} - \vec{n}p = \vec{t}$ *in* $E \subset \Gamma$ *, then*

$$\int_T (\nabla\vec{u} - pI):\nabla \vec{w}_E \, d\Omega = \int_E (\vec{t} - b_0\vec{u})\vec{w}_E \, d\gamma + \int_T \vec{f}.\vec{w}_E \, d\Omega.$$

$$= \int_{\partial T} (\vec{t} - b_0\vec{u})\vec{w}_E \, d\gamma + \int_T \vec{f}.\vec{w}_E \, d\Omega.$$

Theorem 5.6. For any mixed finite element approximation (not necessarily inf-sup stable) defined on rectangular grids T_h, the residual estimator η_R satisfies:

$$\|\vec{e}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \leq C_\Omega \eta_R .$$

$$\eta_{R,T} \leq C \left(\sum_{T' \in \omega_T} \{ \|\vec{e}\|^2_{J,T'} + \|\varepsilon\|^2_{0,T'} \} \right)^{\frac{1}{2}}.$$

Note that the constant C in the local lower bound is independent of the domain, and

$$\|\vec{e}\|^2_{J,T'} = |\vec{e}|^2_{1,T'} + \|\vec{e}\|^2_{0,\partial T'} .$$

Proof. We include this for completeness. To establish the upper bound we let $[\vec{v}, q] \in H^1(\Omega) \times L^2(\Omega)$ and $\vec{v}_h \in X_h$ be the clement interpolant of $\vec{v}$, then

$$B([\vec{e},\varepsilon];[\vec{v},q]) = B([\vec{e},\varepsilon];[\vec{v}-\vec{v}_h,q])$$

$$= \int_\Omega \vec{f}.(\vec{v}-\vec{v}_h)d\Omega - \int_\Omega \nabla\vec{u}_h:\nabla(\vec{v}-\vec{v}_h)d\Omega - \int_\Gamma b_0\vec{u}_h.(\vec{v}-\vec{v}_h)d\gamma + \int_\Gamma \vec{t}.(\vec{v}-\vec{v}_h)d\gamma + \int_\Omega p_h\nabla.(\vec{v}-\vec{v}_h)d\Omega + \int_\Omega q\nabla.\vec{u}_h d\Omega$$

$$= \sum_{T\epsilon T_h} \{ \int_T \vec{f}.(\vec{v}-\vec{v}_h)d\Omega + \int_T (\nabla^2\vec{u}_h - \nabla p_h).(\vec{v}-\vec{v}_h)d\Omega - \sum_{E\epsilon\partial T}\int_\Omega \vec{R}_E(\vec{v}-\vec{v}_h)d\gamma + \int_T q\nabla.\vec{u}_h d\Omega \}.$$

Thus,

$$|B([\vec{e},\varepsilon];[\vec{v},q])| \leq \sum_{T\in T_h} \Big\{ \|\vec{f} + \nabla^2\vec{u}_h - \nabla p_h\|_{0,T} \|\vec{v} - \vec{v}_h\|_{0,T} + \sum_{E\in\partial T} \|\vec{R}_E\|_{0,E} \|\vec{v} - \vec{v}_h\|_{0,E} + \|q\|_{0,T}\|\nabla.\vec{u}_h\|_{0,T} \Big\}$$

Then

$$|B([\vec{e},\varepsilon];[\vec{v},q])| \leq \Big\{ \left(\sum_{T\epsilon T_h} h_T^2 \|\vec{f} + \nabla^2\vec{u}_h - \nabla p_h\|^2_{0,T}\right)^{\frac{1}{2}} \left(\sum_{T\epsilon T_h} \frac{1}{h_T^2}\|\vec{v} - \vec{v}_h\|^2_{0,T}\right)^{\frac{1}{2}} + \left(\sum_{T\in T_h}\sum_{E\in\partial T} h_E\|\vec{R}_E\|^2_{0,T}\right)^{\frac{1}{2}} \left(\sum_{T\in T_h}\sum_{E\in\partial T} \frac{1}{h_E}\|\vec{v} - \vec{v}_h\|^2_{0,E}\right)^{\frac{1}{2}} + \left(\sum_{T\in T_h}\|q\|^2_{0,T}\right)^{\frac{1}{2}} \left(\sum_{T\in T_h}\|\nabla.\vec{u}_h\|^2_{0,T}\right)^{\frac{1}{2}} \Big\}$$

Using lemma 5.3 then gives:

$$|B([\vec{e},\varepsilon];[\vec{v},q])| \leq C\left(\sum_{T\epsilon T_h}\{\|\vec{v}\|^2_{J,T} + \|q\|^2_{0,T}\}\right)^{\frac{1}{2}} \times \left(\sum_{T\epsilon T_h} \left\{ h_T^2\|\vec{R}_T\|^2_{0,T} + \sum_{E\epsilon\partial T} h_E\|\vec{R}_E\|^2_{0,E} + \|R_T\|^2_{0,T} \right\} \right)^{\frac{1}{2}}$$

Finally, using the proposition 3.7 gives:

$$\|\vec{e}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \leq C_\Omega \left(\sum_{T\epsilon T_h} \left\{ h_T^2\|\vec{R}_T\|^2_{0,T} + \sum_{E\epsilon\partial T} h_E\|\vec{R}_E\|^2_{0,E} + \|R_T\|^2_{0,T} \right\} \right)^{\frac{1}{2}}$$

This establishes the upper bound.

Turning to the local lower bound. First, for the element residual part, we have:

$$\int_T \vec{R}_T.\vec{w}_T d\Omega = \int_T (\vec{f} + \nabla^2\vec{u}_h - \nabla p_h).\vec{w}_T d\Omega$$

$$= \int_\Omega \vec{f}.\vec{w}_T d\Omega - \int_\Omega (\nabla\vec{u}_h - p_h I):\nabla \vec{w}_T d\Omega + \int_{\partial T} (\nabla\vec{u}_h - p_h I)\vec{n}.\vec{w}_T d\gamma.$$

Using (41) and (30), gives:

$$\int_T \vec{R}_T.\vec{w}_T d\Omega = \int_T (\nabla\vec{u} - pI):\nabla \vec{w}_T d\Omega - \int_T (\nabla\vec{u}_h - p_h I)\nabla \vec{w}_T d\Omega$$

$$= \int_T (\nabla\vec{e} - \varepsilon I).\nabla \vec{w}_T d\Omega$$

$$\leq \left(|\vec{e}|_{1,T} + \|\varepsilon\|_{0,T}\right)|\vec{w}_T|_{1,T}$$

$$\leq C\left(|\vec{e}|^2_{1,T} + \|\varepsilon\|^2_{0,T}\right)^{\frac{1}{2}} h_T^{-1}\|\vec{R}_T\|_{0,T}.$$

In addition, from the inverse inequality (29),

$$\int_T \vec{R}_T.\vec{w}_T d\Omega = \left\|\vec{R}_T.b_T^{\frac{1}{2}}\right\|^2_{0,T} \geq c\|\vec{R}_T\|^2_{0,T}$$

Thus,

$$h_T^2\|\vec{R}_T\|^2_{0,T} \leq C\left(|\vec{e}|^2_{1,T} + \|\varepsilon\|^2_{0,T}\right)$$

$$\leq C\left(\|\vec{e}\|^2_{J,T} + \|\varepsilon\|^2_{0,T}\right). \quad (44)$$

Next comes the divergence part,

$$\|R_T\|_{0,T} = \|\nabla.\vec{u}_h\|_{0,T} = \|\nabla.(\vec{u} - \vec{u}_h)\|_{0,T}$$

$$\leq \sqrt{2}|\vec{u} - \vec{u}_h|_{1,T}$$

$$\leq \sqrt{2}\|\vec{u} - \vec{u}_h\|_{J,T} = \sqrt{2}\|\vec{e}\|_{J,T} \quad (45)$$

Finally, we need to estimate the jump term. For an edge $E \in \partial T \cap \varepsilon_{h,\Omega}$ we have

$$2\int_E \vec{R}_E.\vec{w}_E d\gamma = \int_{\partial T} (\nabla\vec{u}_h - p_h I)\vec{n}.\vec{w}_E d\gamma$$

$$= \sum_{i=1:2} \int_{\partial T_i} (\nabla\vec{u}_h - p_h I)\vec{n}.\vec{w}_E d\gamma$$

$$= \int_{\omega_E} (\nabla\vec{u}_h - p_h I):\nabla \vec{w}_E d\Omega + \sum_{i=1:2} \int_{T_i} (\nabla^2\vec{u}_h - \nabla p_h).\vec{w}_E d\Omega$$

Using (42) gives:

$$2\int_E \vec{R}_E . \vec{w}_E d\gamma = \int_{\omega_E} (\nabla\vec{u}_h - p_h I) : \nabla \vec{w}_E d\Omega + \sum_{i=1:2} \int_{T_i} (\nabla^2 \vec{u}_h - \nabla p_h) . \vec{w}_E d\Omega - \int_{\omega_E} (\nabla\vec{u} - pI) : \nabla \vec{w}_E d\Omega + \int_{\omega_E} \vec{f} . \vec{w}_E d\Omega$$

$$= \int_{\omega_E} (\nabla\vec{u}_h - p_h I) : \nabla \vec{w}_E d\Omega - \int_{\omega_E} (\nabla\vec{u} - pI) : \nabla \vec{w}_E d\Omega + \sum_{i=1:2} \int_{T_i} \vec{f} . \vec{w}_E d\Omega + \sum_{i=1:2} \int_{T_i} (\nabla^2 \vec{u}_h - \nabla p_h) . \vec{w}_E d\Omega$$

$$\le \left(|\vec{e}|_{1,\omega_E} + \|\varepsilon\|_{0,\omega_E} \right) |\vec{w}_E|_{1,\omega_E} + \sum_{i=1:2} \|\vec{R}_{T_i}\|_{0,T_i} \|\vec{w}_E\|_{0,\omega_E}$$

Using (32) and (33) gives,

$$2\int_E \vec{R}_E . \vec{w}_E d\gamma \le C \left(|\vec{e}|^2_{1,\omega_E} + \|\varepsilon\|^2_{0,\omega_E} \right)^{\frac{1}{2}} h_E^{-\frac{1}{2}} \|\vec{R}_E\|_{0,E} + \sum_{i=1:2} \|\vec{R}_{T_i}\|_{0,T_i} h_E^{\frac{1}{2}} \|\vec{R}_E\|_{0,E}$$

Using (44) gives,

$$2\int_E \vec{R}_E . \vec{w}_E d\gamma \le C \left(\|\vec{e}\|^2_{J,\omega_E} + \|\varepsilon\|^2_{0,\omega_E} \right)^{\frac{1}{2}} h_E^{-\frac{1}{2}} \|\vec{R}_E\|_{0,E} \quad (46)$$

Using (31) gives,

$\int_E \vec{R}_E . \vec{w}_E d\gamma = \left\| \vec{R}_E b_E^{\frac{1}{2}} \right\|^2_{0,E} \ge c \left\| \vec{R}_E \right\|^2_{0,E}$, and thus using (46) gives,

$$h_E \|\vec{R}_E\|^2_{0,E} \le C \left(\|\vec{e}\|^2_{J,\omega_E} + \|\varepsilon\|^2_{0,\omega_E} \right). \quad (47)$$

We also need to show that (47) holds for boundary edges. For an $E \in \partial T \cap \varepsilon_{h,\Gamma}$, we have

$$\int_E \vec{R}_E . \vec{w}_E d\gamma = \int_{\partial T} \left[(b_0 \vec{u}_h + (\nabla \vec{u}_h - p_h I)\vec{n} - \vec{t} \right] . \vec{w}_E \, d\gamma$$

$$= \int_{\partial T} (b_0 \vec{u}_h - \vec{t}) . \vec{w}_E d\gamma + \int_{\partial T} ((\nabla \vec{u}_h - p_h I) \vec{n} . \vec{w}_E d\gamma$$

$$= \int_{\partial T} (b_0 \vec{u}_h - \vec{t}) . \vec{w}_E d\gamma + \int_T ((\nabla \vec{u}_h - p_h I) : \nabla \vec{w}_E d\Omega + \int_T (\nabla^2 \vec{u}_h - \nabla p_h I) . \vec{w}_E \, d\Omega .$$

Using (43) and (31), gives

$$\int_E \vec{R}_E . \vec{w}_E d\gamma = \int_{\partial T} (b_0 \vec{u}_h - \vec{t}) . \vec{w}_E d\gamma + \int_T ((\nabla \vec{u}_h - p_h I) : \nabla \vec{w}_E d\Omega + \int_T (\nabla^2 \vec{u}_h - \nabla p_h I) . \vec{w}_E d\Omega$$

$$= -\int_T ((\nabla \vec{e} - \varepsilon I) : \nabla \vec{w}_E \, d\Omega - \int_{\partial T} b_0 \vec{e} . \vec{w}_E \, d\gamma + \int_T \vec{R}_T . \vec{w}_E \, d\Omega$$

$$\le C \left(\|\vec{e}\|_{J,T} + \|\varepsilon\|_{0,T} \right) \|\vec{w}_E\|_{J,T} + \|\vec{R}_T\|_{0,T} h_E^{\frac{1}{2}} \|\vec{R}_E\|_{0,E} .$$

Using (44) and (34), we obtain

$$\int_E \vec{R}_E . \vec{w}_E d\gamma \le C \left(\|\vec{e}\|^2_{J,T} + \|\varepsilon\|^2_{0,T} \right)^{\frac{1}{2}} h_E^{-\frac{1}{2}} \|\vec{R}_E\|_{0,E} . \quad (48)$$

Using (31)

$$\int_E \vec{R}_E . \vec{w}_E d\gamma = \left\| \vec{R}_E b_E^{\frac{1}{2}} \right\|^2_{0,E} \ge c \left\| \vec{R}_E \right\|^2_{0,E},$$

and thus using (48) gives,

$$h_E \|\vec{R}_E\|^2_{0,E} \le C \left(\|\vec{e}\|^2_{J,T} + \|\varepsilon\|^2_{0,T} \right). \quad (49)$$

Finally, combining (44), (45), (47) and (49) establishes the local lower bound.

Remark 5.7. Theorem 5.6 also holds for stable (and unstable) mixed approximations defined on a triangular subdivision if we take the obvious interpretation of ω_T. The Proof is identical except for the need to define appropriate element and edge bubble functions.

VI. FIGURES AND TABLES

In this section some numerical results of calculations with mixed finite element Method and ADINA System will be presented. Using our solver, we run the test problem driven cavity flow [14, 16] with a number of different model parameters.

Example. Square domain, enclosed flow boundary condition.

This is a classic test problem used in fluid dynamics, known as driven-cavity flow. It is a model of the flow in a square cavity with the lid moving from left to right. Let the computational model:

($y = 1; -1 \le x \le 1$ / $u_x = 1 - x^2$), a regularized cavity. With these data, see that the (D+N) condition is satisfied, just take b_0 a real number very large and $\vec{t} = (b_0(1 - x^2); 0)$ on Γ_1 =($y = 1; -1 \le x \le 1$) and $\vec{t} = (0; 0)$ on the other three boundary of the square domain.

The streamlines are computed from the velocity solution by solving the Poisson equation numerically subject to a zero Dirichlet boundary condition.

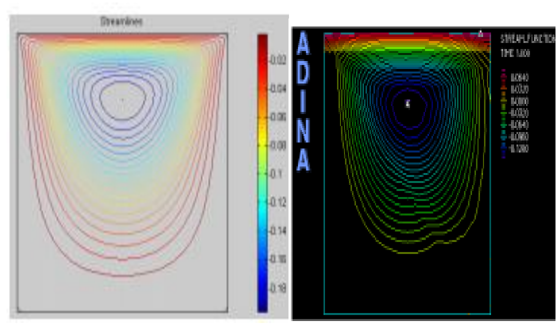

Fig.1. Uniform streamline plot by MFE (left) associated with a 64-64 square grid, Q_1-P_0 approximation, and uniform streamline plot (right) computed with ADINA system.

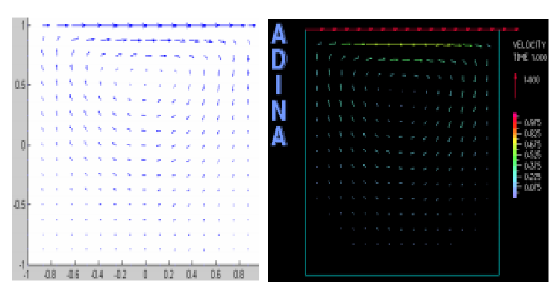

Fig.2. Velocity vectors solution by MFE (left) associated with a 64-64 square grid, $Q_1.P_0$ approximation and velocity vectors solution (right) computed with ADINA system.

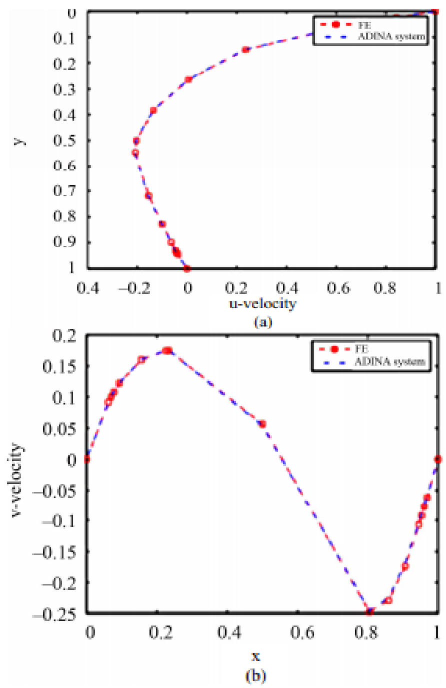

Fig.3. The velocity component u at vertical center line (a), and the velocity component v at horizontal center line (b) with a 129 × 129 grid.

Figure 3 shows the velocity profiles for lines passing through the geometric center of the cavity.

These features clearly demonstrate the high accuracy achieved by the proposed mixed finite element method for solving the Stokes equations in the lid-driven squared cavity.

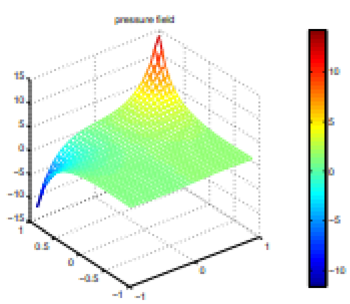

Fig.4. Pressure plot for the flow with a 64-64 square grid.

VII. CONCLUSION

In this work, we were interested in the numerical solution of the partial differential equations by simulating the flow of an incompressible fluid. We introduced the Stokes equations with a boundary condition of type (D+N). The weak formulation obtained is a problem of saddle point type. We have shown the existence and uniqueness of the solution of this problem. We used the discretization by mixed finite element method with a posteriori error estimation of the computed solutions. For the test of driven-cavity flow, the particles in the body of the fluid move in a circular trajectory.

Our results agree with Adina system.

Numerical results are presented to see the performance of the method, and seem to be interesting by comparing them with other recent results.

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