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Applied lacunary interpolation for solving Boundary value problems

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Abstract
Interpolation by various types of splines is the standard procedure in many applications. In this paper shall discuss the function, two and fourth derivatives of spline interpolation as an alternative to polynomial spline interpolation at the all intervals. The method is appropriate and solving of initial and boundary value problems, the results revealing that method is very effective and accurate.

Keywords: Cauchy problem, Spline function, Initial and Boundary value problems, Taylor's expansion.

1. Introduction.

We consider the following initial and boundary values problem:

$$y''(x) = f(x, y(x), y'(x)),\ x \in [0, 1],\ \ y(a) = y_1,\ y'(a) = y'_2$$
$$y''(x) = f(x, y(x), y'(x)),\ x \in [0, 1],\ \ y(a) = y_3,\ y'(b) = y'_4 \qquad (1)$$

With the help of lacunary spline functions of type (0, 2, 4) see Faraidun (2010) [2], by using that $f \in C^{n-1}([0,1] \times R^2)$, $n \geq 2$ and that it satisfies the Lipschitz continuous

$$\left| f^{(q)}(x, y_1, y'_1) - f^{(q)}(x, y_2, y'_2) \right| \leq L\{|y_1 - y_2| + |y'_1 - y'_2|\},\ q=0,1,\ldots, n\text{-}1 \qquad (2)$$

Also boundary value problems are satisfied, and for all $x \in [0,1]$ and for all real $y_1,\ y_2,\ y'_1,\ y'_2$. These conditions ensure the existence of unique solution of the problem (1).

In [2] authors investigated the model (0, 2, 4) approximation by polynomial splines on box partitions in all intervals. The main computational advantage of this technique is its simple applicability for solving boundary value problems. We develop a new spline approximation method for solving the boundary value problems over the interval [a, b].

In section 2, we give a brief description of the method. The derivation of the difference schemes spline function has been given in Section 3, and also, we have shown the second-order accuracy method and convergence analysis are studied. We have solved two numerical examples to demonstrate the applicability of the methods in section 4. In the last section, the discussion on the results is given in Section 5.

2. Construct of approximate values:

Let $\omega(h,\ y^{(r)}) = \underset{|x-x_1|\leq h}{Max}\left\{\left|y^{(r)}(x) - y^{(r)}(x)\right|\right\}$, $r = 0,1, \ldots.,6.$ And let $\overline{Y}_k^{(q)}$: $\bar{y}_0^{(q)}, \bar{y}_1^{(q)}, \bar{y}_2^{(q)}, \ldots, \bar{y}_n^{(q)}$; $q = 0,1, \ldots.,6,$ be approximate to the exact values $Y_k^{(q)}$: $y_0^{(q)}, y_1^{(q)}, y_2^{(q)}, \ldots, y_n^{(q)}$; $q = 0,1, \ldots.,6.$

Now from these approximate values we construct a spline function $\overline{S}_\Delta(x)$ wich interpolates to the set $\overline{Y}$ on the mesh Δ and approximate the solution y(x) of equation (1) as [4, 5]. The set $\overline{Y}^{(q)}$ is defined as:

$$\bar{y}_0 = y_0,\ \bar{y}'_0 = y'_0,\ \bar{y}_0^{(2+q)} = f^{(q)}(x_0,\ y_0\ \ y'_0) \text{ where } q = 0\ 1\ldots, r.$$

$$\bar{y}_{k+1} = \bar{y}_k + h\bar{y}'_k + \int_{x_k}^{x_{k+1}} \int_{x_k}^{t} f[u,\ y_k^*(u),\ y_k^{**}{}'(u)]du\,dt,$$

$$\bar{y}'_{k+1} = \bar{y}'_k + \int_{x_k}^{x_{k+1}} f[t,\ y_k^*(t),\ y_k^{**}{}'(t)]dt,$$

$$\bar{y}_{k+1}^{(q+2)} = f^{(q)}(x_{k+1}, \bar{y}_{k+1}, \bar{y}'_{k+1}),\ q = 0,2,4.,,\ k = 0,1,2,....,m-1$$

and for $x_k \le x \le x_{k+1}$

$$y_k^*(x) = \sum_{j=0}^{r+2} (x - x_k)^j \frac{\bar{y}_k^{(j)}}{j!}, \qquad y_k^*{}'(x) = \sum_{j=0}^{r+1} (x - x_k)^j \frac{\bar{y}_k^{(j+1)}}{j!}$$

and $$y'^{**}_{k+1}(x) = \bar{y}'_k + \int_{x_k}^{x_1} f[t,\ y_k^*(t),\ y_k^*{}'(t)]dt$$

Using these approximate values $\bar{Y}_k^{(q)}$ $(q = 0,2,4.,\ k = 0,1,2,....,m)$ and $\bar{y}_0'$, $\bar{y}_m'$ on the bases of [2, 3], we construct the lacunary spline function $\bar{S}_\Delta(x)$ of the type (0, 2, 4) ($\bar{S}_\Delta(x) = \bar{S}_k(x)$ if $x_k \le x \le x_{k+1}$) and denote by $\bar{S}_{n,6}^5$ the class of six degree splines $\bar{S}(x)$ as the following:

$$G(x) = \begin{cases} \bar{S}_\Delta(x_k) = \bar{y}_k \\ \bar{S}_\Delta^{(q)}(x_k) = \bar{y}_k^{(q)} \end{cases} \tag{3}$$

Where $q = 2,\ 4$ and $k = 0,1,2,....,m$, the existence and uniqueness of the above spline function have been shown in [2],

$$\bar{S}_0 = \bar{y}_0 + (x - x_0)\bar{y}'_0 + \frac{(x-x_0)^2}{2}\bar{y}''_0 + \bar{a}_{0,3}\bar{y}_o^{(3)} + \frac{(x-x_0)^4}{24}\bar{y}_o^{(4)} + (x-x_0)^5\bar{a}_{0,5} + (x-x_0)^6\bar{a}_{0,6} \tag{4}$$

Let us examine now intervals $[x_i, x_{i+1}]$, *i*=1,2,…, *n*-2., Defined $\bar{S}_i(x)$ as:

$$\bar{S}_i(x) = \bar{y}_i + (x-x_i)\bar{a}_{i,1} + \frac{(x-x_i)^2}{2}\bar{y}''_i + (x-x_i)^3\bar{a}_{i,3} + \frac{(x-x_i)^4}{24}\bar{y}_i^{(4)} + (x-x_i)^5\bar{a}_{i,5} + (x-x_i)^6\bar{a}_{i,6} \tag{5}$$

Here

$$\bar{a}_{0,3} = \frac{5h^{-3}}{3}(\bar{y}_1 - \bar{y}_0) - \frac{1}{18h}(2\bar{y}''_1 + 13\bar{y}''_0) - \frac{5}{3}h\ \bar{y}'_0 + \frac{h}{216}(\bar{y}_1^{(4)} - 4\bar{y}_0^{(4)});$$

$$\bar{a}_{0,5} = h^{-5}(\bar{y}_0 - \bar{y}_1) + \frac{h^{-3}}{6}(\bar{y}''_1 + 2\bar{y}''_0) + h^{-4}\bar{y}'_0 - \frac{h^{-1}}{360}(4\bar{y}_1^{(4)} + 11\bar{y}_0^{(4)});$$

and

$$\bar{a}_{0,6} = \frac{h^{-6}}{3}(\bar{y}_1 - \bar{y}_0) - \frac{h^{-4}}{18}(\bar{y}''_1 + 2\bar{y}''_0) - \frac{h^{-5}}{3}\bar{y}''_0 + \frac{h^{-2}}{1080}(7\bar{y}_1^{(4)} + 8\bar{y}_0^{(4)});$$

Also

$$\bar{a}_{i,1} + \bar{a}_{i+1,1} = 2h^{-1}(\bar{y}_{i+1} - \bar{y}_i) + \frac{h}{6}(\bar{y}''_{i+1} - \bar{y}''_i) - \frac{h^3}{360}(\bar{y}_{i+1}^{(4)} - \bar{y}_i^{(4)});$$

$$\bar{a}_{i,3} = -\frac{5}{3}h^{-1}\bar{a}_{i,1} + \frac{5}{3}h^{-3}(\bar{y}_{i+1} - \bar{y}_i) - \frac{1}{18}h^{-2}(2\bar{y}''_{i+1} + 13\bar{y}''_i) + \frac{h}{216}(\bar{y}_{i+1}^{(4)} - 4\bar{y}_i^{(4)});$$

$$\bar{a}_{i,5} = h^{-4}\bar{a}_{i+1,1} - h^{-5}(\bar{y}_{i+1} - \bar{y}_i) + \frac{h^{-3}}{6}(\bar{y}''_{i+1} + 2\bar{y}''_i) - \frac{h^{-1}}{360}(4\bar{y}_{i+1}^{(4)} + 11\bar{y}_i^{(4)});$$

and

$$\bar{a}_{i,6} = -\frac{h^{-5}}{3}\bar{a}_{i,1} + \frac{h^{-6}}{3}(\bar{y}_{i+1} - \bar{y}_i) - \frac{h^{-4}}{18}(\bar{y}''_{i+1} + 2\bar{y}''_i) + \frac{h^{-2}}{1080}(7\bar{y}_{i+1}^{(4)} + 8\bar{y}_i^{(4)}).$$

Similarly for the last interval $[x_n,\ x_{n-1}]$, we can define approximate values of $\bar{S}_n(x)$.

3. Convergence of a spline functions to a solution:

A key ingredient in the development of our estimates is the following theorem which gives a bound on the size of a polynomial on a spline function $\bar{S}_\Delta(x)$ in terms of its values on a discrete subset which is scattered in the values of $y_k\,(k=0,1,2,\ldots,m)$ of a problem (1).

Theorem 1: Let $\bar{y}_k^{(q)}$ $(q=0,2,4; k=0,1,2\ldots,m)$ be the approximate values defined above. Then the following estimates of spline function $\bar{S}_\Delta(x)$ are valid:

(i) $\left|S_k^{(q)}(x)-\bar{S}_k^{(q)}(x)\right| \le CI_q\, h^{6-q}\,\omega_6(h)$; for $q=0,1,\ldots,6,\ k=0,1,\ldots,m-2$

where C_q denote the difference constants dependent of h.

(ii) $\left|y_k^{(q)}(x)-\bar{S}_k^{(q)}(x)\right| \le H_q \sum_{j=0}^{q-1} \theta^{-j}(h_j + h^j \left\|D^j f\right\|_p)$; for $q=0,1,\ldots,6$, where $y(x)$ is a solution of problem (1) and D_q denote the difference constants dependent of h.

Proof: (i) From theorem 1 of [1] and equation (3), we have

$$S_0(x)-\bar{S}_0(x) = (x-x_0)^3\,(a_{0,3}-\bar{a}_{0,3}) + (x-x_0)^5\,(a_{0,5}-\bar{a}_{0,5}) + (x-x_0)^6\,(a_{0,6}-\bar{a}_{0,6}) \qquad (6)$$

Where

$$a_{0,3}-\bar{a}_{0,3} = \frac{5}{3h^3}(y_1-\bar{y}_1) - \frac{1}{9h}[y_1''-\bar{y}_1''] + \frac{h}{216}[y_1^{(4)}-\bar{y}_1^{(4)}]$$

implies that

$$\left|a_{0,3}-\bar{a}_{0,3}\right| \le \frac{1}{216}(C_1+24C_2+72C_3)\ \omega_6(h) = \frac{1}{216} I_1\ \omega_6(h)$$

where $I_1 = C_1 + 24C_2 + 72C_3$ and C_1 , C_2 and C_3 are constants dependent of h.

Similarly

$$\left|a_{0,5}-\bar{a}_{0,5}\right| \le \frac{1}{h^5}\left|y_1-\bar{y}_1\right| + \frac{1}{6h^3}\left|y_1''-\bar{y}_1''\right| + \frac{1}{90h}\left|y_1^{(4)}-\bar{y}_1^{(4)}\right|$$

$$\le \frac{1}{90}(C_4+15C_5+90C_6)\omega_6(h) = \frac{1}{90} I_2\omega_6(h)$$

where $I_2 = C_4 + 15C_5 + 90C_6$ and C_4 , C_5 and C_6 are constants dependent of h.

and

$$\left|a_{0,6}-\bar{a}_{0,6}\right| \le \frac{1}{3h^6}\left|y_1-\bar{y}_1\right| + \frac{1}{18h^4}\left|y_1''-\bar{y}_1''\right| + \frac{7}{1080h^2}\left|y_1^{(4)}-\bar{y}_1^{(4)}\right|$$

$$\le \frac{1}{1080}(7C_7+60C_8+360C_9)\omega_6(h) = \frac{1}{1080} I_3\omega_6(h)$$

where $I_3 = 7C_7 + 60C_8 + 360C_9$ and C_7 , C_8 and C_9 are constants dependent of h.

And hence

$$\left|S_0(x)-\bar{S}_0(x)\right| \le h^3\left|a_{0,3}-\bar{a}_{0,3}\right| + h^5\left|a_{0,5}-\bar{a}_{0,5}\right| + h^6\left|a_{0,6}-\bar{a}_{0,6}\right|$$

$$\le I\ \omega_6(h)$$

Where $I = I_1 + I_2 + I_3$, dependent of h.

By taking the first derivative of equation (5), we have

$$\left|S_0'(x)-\bar{S}_0'(x)\right| \le \frac{2}{h}\left|y_1-\bar{y}_1\right| + \frac{5h}{6}\left|y_1''-\bar{y}_1''\right| - \frac{h^3}{360}\left|y_1^{(4)}-\bar{y}_1^{(4)}\right|$$

$$\le \frac{1}{360}(\bar{C}_1 + 300\bar{C}_2 + 720\bar{C}_3)\,\omega_6(h) = \frac{1}{360}\bar{I}_1\omega_6(h)$$

and by successive differentiations obtain

$$\left|S_0^{(q)}(x)-\bar{S}_0^{(q)}(x)\right| \le I_q\, h^{6-q}\,\omega_6(h); \text{for } q=0,1,\ldots,6.$$

This proves (i) for $k=0$ and $x\in[x_0, x_1]$. Further more in the interval $[x_{k-1}, x_k]$

$$S_k(x)-\bar{S}_k(x) = (x-x_k)(a_{k,1}-\bar{a}_{k,1}) + (x-x_k)^3(a_{k,3}-\bar{a}_{k,3}) + (x-x_k)^5(a_{k,5}-\bar{a}_{k,5})^5$$
$$+(x-x_k)^6(a_{k,6}-\bar{a}_{k,6})$$

From [2, 6], it's clear that, to show

$$a_{k,1}-\bar{a}_{k,1} = \frac{2}{h^2}(y_1-\bar{y}_1) + \frac{h}{6}(y_1''-\bar{y}_1'') + \frac{h^3}{360}[y_1^{(4)}-\bar{y}_1^{(4)}]$$

implies that

$$\left|a_{k,2}-\bar{a}_{k,2}\right| \le \frac{1}{360}(2C_0^* + 6C_1^* + C_2^*)\,\omega_6(h) = \frac{1}{360}I_1^*\,\omega_6(h)\;;$$

where I_1^* and C_0^*, C_1^* and C_2^* be a constants dependent of h.

Similarly

$$\left|a_{k,3}-\bar{a}_{k,3}\right| \le \frac{5}{3h}\left|\bar{a}_{k,1}-a_{k,1}\right| + \frac{5}{3h^3}\left|y_{k+1}-\bar{y}_{k+1}\right| + \frac{1}{9h^2}\left|\bar{y}_{k+1}^{(2)}-y_{k+1}^{(2)}\right| + \frac{h}{216}\left|y_{k+1}^{(4)}-\bar{y}_{k+1}^{(4)}\right|$$

$$\le \frac{1}{216}(360C_3^* + 360C_4^* + 24C_5^* + C_6^*)\,\omega_6(h) = \frac{1}{216}I_2^*\,\omega_6(h)$$

where I_1^* and C_3^*, C_4^*, C_5^* and C_6^* be a constants dependent of h.

And also

$\left|a_{k,5}-\bar{a}_{k,5}\right| \le I_3^*\,\omega_6(h)$; $\left|a_{k,6}-\bar{a}_{k,6}\right| \le I_4^*\,\omega_6(h)$, where I_2^* and I_4^* are dependent of h.

and by taking the successive differentiation, we obtain

$\left|S_k^{(q)}(x)-\bar{S}_k^{(q)}(x)\right| \le I_q\, h^{6-q}\,\omega_6(h);$ for $q=0,1,\ldots,6.$ Which is prove (i) for $k=0,1,\ldots,m-2.$

We can repeat the same manner in above for $k=m-1$.

Proof of theorem 1 (ii):

$$\left|y^{(q)}(x)-\bar{S}_\Delta^{(q)}(x)\right| \le C\left(\left\|y^{(q)}(x)-S_\Delta^{(q)}(x)\right\|_{L\infty} + \left\|S_\Delta^{(q)}(x)-\bar{S}_\Delta^{(q)}(x)\right\|_{L\infty}\right)$$

From theorem 2 [4], and after some derivations the following estimates are valid

$$\left\|y^{(q)}(x)-\bar{S}_\Delta^{(q)}(x)\right\|_{L\infty} \le C_q h^{6-q}\omega_6(h) \text{ , where } h=\theta\Delta c \text{ and } \omega_6(f;h)_p \le C\sum_{j=0}^{r-1}\theta^{-j}(h_r + h^r\left\|D^r f\right\|_p) \qquad (7)$$

Using equation (7) and estimate in (i), we have

$$\left|y^{(q)}(x)-\bar{S}_\Delta^{(q)}(x)\right| \le C_q h^{6-q}\omega_6(h) + I_q h^{6-q}\omega_6(f;h)$$

$$= (C_q + I_q)h^{6-q}\omega_6(f;h) = H_q h^{6-q}\omega_6(f;h) \le H_q\sum_{j=0}^{q-1}\theta^{-j}(h_j + h^j\left\|D^j f\right\|_p)$$

Which is proves (ii).

Theorem 2: If the function f in Cauchy's problem (1) satisfies conditions (2) and (3), then the following inequalities are hold:

$\left\|\bar{S}_0''(x)-f[x,\bar{S}_0(x),\bar{S}_0'(x)]\right\|_{Lp}\le I_{0,2}^{*}\,\omega_6(h)$ where $I_{0,2}^{*}$ is constants dependent of h and $x\in[x_0,\,x_1]$,

$\left\|\bar{S}_k''(x)-f[x,\bar{S}_k(x),\bar{S}_k'(x)]\right\|_{Lp}\le I_{k,2}^{*}\,\omega_6(h)$ where $I_{k,2}^{*}$ is constants dependent of h and $x\in[x_{k-1},\,x_k]$,

$\left\|\bar{S}_{m-1}''(x)-f[x,\bar{S}_{m-1}(x),\bar{S}_{m-1}'(x)]\right\|_{Lp}\le I_{m-1,2}^{*}\,\omega_6(h)$ where $I_{m-1,2}^{*}$ is constants dependent of h and $x\in[x_{m-1},\,x_m]$.

Proof: Using condition (1), (2) and (3), we have

$\left\|D^q(f(x)-y(x))\right\|_{Lp}\le C_1\omega_q(f;b-a)$ and $\left\|D^q y(x)\right\|_{Lp}\le C_2\,\omega_q(f;1)$

by the Taylor expanssion of y about zero, then

$$\left|D^q(y(x)-\bar{S}_\Delta(x))\right|\le\int_0^u\left|D^{q+1}(y-\bar{S}_\Delta)(u)\right|du\le\left\|D^{q+1}(y-\bar{S}_\Delta)\right\|_{Lp}$$

$$\le 2^{-\frac{r}{p}}\left\|D^q y\right\|_{Lp}\le C_3\,\omega_q(f;1)$$

$$\left\|D^q(\bar{S}_0(x)-f(x))\right\|_{Lp}\le\left\|D^q(\bar{S}_0-y)\right\|_{Lp}+\left\|D^q y-f\right\|_{Lp}\le I_{0,2}^{*}\omega_6(f;1)\text{; where } q=2.$$

Similarly for each the intervals can be proving it.

4. Numerical results:

In this section, the method discussed in section 2 and 3 were tested on two problems, and the absolute errors in the analytical solution were calculated. Our results confirm the theoretical analysis of the methods with the initial and boundary value problems. For different starting points observed same convergence point with or less iterations, see [7].

Problem (1): we consider that the second order boundary value problem $y''+y=0$ where $x\in[0,1]$ and $y(0)=1,\ y'(0)=1$.

Problem (2): Let $y'''-y'=2\cos(x)$ where $y(0)=3, y'(0)=2, y'(1)=2$.

It turns out that the six degree spline which presented in this paper, yield approximate solution that is O(h^6) as stated in Theorem 1. The results are shown in the Table 1 and Table 2 for different step sizes h.

Table 1 Absolute maximum error for the derivatives $\bar{S}(x)$.

h	$\left\|\bar{s}'(x)-y'(x)\right\|_\infty$	$\left\|\bar{s}'''(x)-y'''(x)\right\|_\infty$	$\left\|\bar{s}^{(5)}(x)-y^{(5)}(x)\right\|_\infty$	$\left\|\bar{s}^{(6)}(x)-y^{(6)}(x)\right\|_\infty$
0.1	67.67×10^{-10}	26.06×10^{-7}	73×10^{-4}	11.33×10^{-2}
0.01	64.71×10^{-16}	22.41×10^{-11}	72.05×10^{-6}	11.1×10^{-3}
0.001	44.04×10^{-14}	22.2×10^{-7}	26.64×10^{0}	53.2×10^{3}

Table 2 Absolute maximum error for the derivatives $\bar{S}(x)$.

h	$\left\|\bar{s}'(x)-y'(x)\right\|_\infty$	$\left\|\bar{s}'''(x)-y'''(x)\right\|_\infty$	$\left\|\bar{s}^{(5)}(x)-y^{(5)}(x)\right\|_\infty$	$\left\|\bar{s}^{(6)}(x)-y^{(6)}(x)\right\|_\infty$
0.1	69×10^{-8}	25.32×10^{-5}	50.33×10^{-2}	82.005×10^{-1}
0.01	72.62×10^{-13}	25.38×10^{-8}	53.4×10^{-3}	87.01×10^{-1}
0.001	66.61×10^{-18}	22.2×10^{-11}	5×10^{-3}	79×10^{-1}

5. Conclusion:

An efficient and accurate numerical scheme based on the Interpolation method proposed for solving initial and boundary value problems. The Lacunary interpolation method was employed to reduce the problem to the solution of differential equations. Illustrative examples are presented in Table 1 and 2, were given to demonstrate the validity and applicability of the method with the less errors bounded.

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