A novel traveling wave solution for the generalized combined KdV and mKdV equation using the generalized $\left(\frac{G'}{G}\right)$-expansion method

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ABSTRACT

In this paper, the generalized $\left(\frac{G'}{G}\right)$-expansion method is used for construct an innovative explicit traveling wave solutions involving parameters of the generalized combined KdV and mKdV equation $u_t + a(t)u_x + b(t)u\,u_x + c(t)u^2 u_x + e(t)u_{xxx} = 0$, for any arbitrary functions $a(t), b(t), c(t)$ and $e(t)$. The properties of this method is that gives an explicit solutions other than the other methods and also gives especial solutions.

Keywords **- Combined KdV and mKdV equation, Generalized $\left(\frac{G'}{G}\right)$- expansion method, Traveling wave solutions.**

I. INTRODUCTION

Phenomena in physics and other fields are often described by nonlinear evolution equations (NLEEs).When we want to understand the physical mechanism of phenomena in nature, described by nonlinear evolution equations, exact solutions for the nonlinear evolution equations have to be explored. For example, the wave phenomena observed in fluid dynamics [1,2], plasma and elastic media [3,4] and optical fibers [5,6], etc. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been proposed, such as Hirota's bilinear method [7], Backlund transformation [8], Painlevé expansion [9], sine--cosine method [10], homogeneous balance method [11], homotopy perturbation method [12--14], variational iteration method [15--18], asymptotic methods [19], non-perturbative methods [20], Adomian decomposition method [21], tanh-function method [22- 26], algebraic method [27-30], Jacobi elliptic function expansion method [31- 33], F-expansion method [34--36] and auxiliary equation method [37- 40]. Recently, Wang et al. [41] introduced a new direct method called the $\left(\frac{G'}{G}\right)$-expansion method to look for travelling wave solutions of NLEEs. Nofal et al. [45] used the improved $\left(\frac{G'}{G}\right)$- expansion method for construct explicit the traveling wave solution involving parameters of the fifth- order KdV equation. Hamed et al. [46] introduced the improved $\left(\frac{G'}{G}\right)$- expansion method for construct explicit the traveling wave solutions involving parameters of the $(3+1)-$dimensional Potential equation. Elagan et al. [47] used the generalized $\left(\frac{G'}{G}\right)$- expansion method for construct an innovative explicit traveling wave solutions involving parameters of the Fitz Hugh-Nagumo equation. Hamed et al. [48] used the generalized $\left(\frac{G'}{G}\right)$-expansion method for construct an innovative explicit traveling wave solution involving parameter of the generalized combined KdV and mKdV equation.

Consider the generalized combined KdV and mKdV equation

$$u_t + a(t)u_x + b(t)u\,u_x + c(t)u^2 u_x + e(t)u_{xxx} = 0, \tag{1.1}$$

where $a(t), b(t), c(t)$ and $e(t)$ are functions of t. Eq. (1.1) where $a(t) = 0$ and $b(t), c(t), e(t)$ are constants has been widely used in many physical fields such as plasma physics, fluid physics,solid-state physics and quantum field theory. When $a(t)=c(t)=0$, and $b(t), e(t)$ are constants Eq. (1.1) becomes KdV equation. When $a(t)=b(t)=0$, and $c(t), e(t)$ are constants Eq. (1.1) is mKdV equation. KdV equation and mKdV equation had been studied by many authors. Recently, Zhang [42] obtained some exact solutions of Eq. (1.1) where $a(t)=0$, and $b(t), c(t), e(t)$ are constants by tanh function method and the direct method. Recently Eq. (1.1) has solved by Zhenya [43] using a generalized approch based on Riccati equation when and $a(t), b(t), c(t), e(t)$ are all constants. In this paper we try to solve Eq.(1.1) using generalized $\left(\frac{G'}{G}\right)$-expansion method when $a(t), b(t), c(t)$ and $e(t)$ are functions of t. The $\left(\frac{G'}{G}\right)$- expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, and that $G=G(\xi)$ satisfies a second order linear ordinary differential equation (LODE):

$G''+\lambda G'+\mu G=0$, where $G'=\frac{dG(\xi)}{d\xi}, G''=\frac{d^2G(\xi)}{d\xi^2}, \xi=x-Vt$, V is a constant. The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given NLEE. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. By using the $\left(\frac{G'}{G}\right)$-expansion method, Wang et al. [41] successfully obtained more travelling wave solutions of four NLEEs. Very recently, Zhang et al. [44] proposed a generalized $\left(\frac{G'}{G}\right)$-expansion method to improve the work made in [41]. The main purpose of this paper is to use generalized $\left(\frac{G'}{G}\right)$-expansion method to solve the generalized FitzHugh-Nagumo equation. The performance of this method is reliable, simple and gives many new solutions, its also standard and computerizable method which enable us to solve complicated nonlinear evolution equations in mathematical physics. The paper is organized as follows. In Section 2, we describe briefly the generalized $\left(\frac{G'}{G}\right)$-expansion method, where $G=G(\xi)$ satisfies the second order linear ordinary differential equation

$$G''+\lambda G'+\mu G=0, \xi=p(t)x+q(t).$$

In Section 3, we apply this method to the generalized Mkdv equation. In section 4, some conclusions are given.

II. DESCRIPTION THE GENERALIZED $\left(\frac{G'}{G}\right)$-EXPANSION METHOD

Suppose that we have the following nonlinear partial differential equation

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots)=0, \tag{2.1}$$

we suppose its solution can be expressed by a polynomial $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi)=\sum_{i=1}^{n}\alpha_i(t)\left(\frac{G'}{G}\right)^i+\alpha_0(t), \quad \alpha_j(t)\neq 0, \tag{2.2}$$

where $\alpha_0(t)$ and $\alpha_j(t)$ are functions of t $(j=1,2,\ldots,n)$ and $\xi=\xi(x,t)$ is a function of x,t to be determine later, $G=G(\xi)$ satisfies the second order linear ordinary differential equation

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0, \tag{2.3}$$

To determine u explicitly we take the following four steps.

Step 1. Determine the integer n by balancing the highest order nonlinear terms and the highest order partial derivative of u in Eq. (2.1).

Step 2. Substitute Eq.(2.2) along with Eq. (2.3) into Eq. (2.1) and collect all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left hand side of Eq. (2.1) is converted into a polynomial in $\left(\frac{G'}{G}\right)$. Then set each coefficient of this polynomial to zero to derive a set of over-determined partial differential equations for $\alpha_0(t)$, $\alpha_i(t)$ and ξ.

Step 3. Solve the system of all equations obtained in step 2 for $\alpha_0(t)$, $\alpha_i(t)$ and ξ by use of Maple.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions of Eq. (2.3) depending on $\left(\frac{G'}{G}\right)$, since the solutions of this equation have been well known for us, then we can obtain exact solutions of Eq. (2.1).

III. THE GENERALIZED COMBINED KDV AND MKDV EQUATION

In this section, we apply the generalized $\left(\frac{G'}{G}\right)$- expansion method to solve the generalized FitzHugh-Nagumo equation, construct the traveling wave solutions for it as follows:

Let us first consider the generalized KdV equation

$$u_t + a(t)u_x + b(t)u\,u_x + c(t)u^2 u_x + e(t)u_{xxx} = 0, \tag{3.1}$$

where $a(t), b(t), c(t)$ and $e(t)$ are functions of t. There is no any method gvae the exact solution of the above equation before. In order to look for the traveling wave solutions of Eq. (3.1) we suppose that

$$u(x,t) = u(\xi), \xi = p(t)x + q(t) \tag{3.2}$$

Suppose that the solution of Eq. (3.1) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows

$$u(\xi) = \sum_{i=1}^{n} \alpha_i(t)\left(\frac{G'(\xi)}{G(\xi)}\right)^i + \alpha_0(t) \tag{3.3}$$

considering the homogeneous balance between u_{xxx} and $u^2 u_x$ in Eq. (3.1) we required that $n+3 = 2n+n+1$, then $n=1$. So we try to find a solution of the form

$$u(t,x) = \alpha_0(t) + \alpha_1(t)\frac{G'(\xi)}{G(\xi)}, \tag{3.4}$$

where G satisfies

$$G' + \lambda G' + \mu G = 0.$$

It is easy to see that $p(t)$ must be a constant function assuming that $\alpha_1(t)$ is not zero on any interval of positive length. Substituting Eq.(3.4) into Eq.(3.1) along with Eq.(2.3) and comparing the coefficients of $\left(\frac{G'}{G}\right)^k, k = 0,1,2,3,4$ we obtain the following equations

$$\alpha_0' = \alpha_1\mu\left(q' + ap + b\alpha_0 p + c\alpha_0^2 p + ep^3\left(2\mu + \lambda^2\right)\right) \tag{3.5}$$

$$\alpha_1' = \alpha_1\left(q'\lambda + ap\lambda + b\alpha_0 p\lambda + b\alpha_1 p\mu + c\alpha_0^2 p\lambda + 2c\alpha_0\alpha_1 p\mu + ep^3\lambda\left(8\mu + \lambda^2\right)\right) \tag{3.6}$$

$$-\alpha_1 q' = \alpha_1 p\left(a + b\alpha_0 + b\alpha_1\lambda + c\alpha_0^2 + 2c\alpha_0\alpha_1\lambda + c\alpha_1^2\mu + ep^2\left(8\mu + 7\lambda^2\right)\right) \tag{3.7}$$

$$-\alpha_1 p\left(b\alpha_1 + 2c\alpha_0\alpha_1 + c\alpha_1^2\lambda + 12ep^2\lambda\right) = 0 \tag{3.8}$$

$$-\alpha_1 p\left(c\alpha_1^2 + 6ep^2\right) = 0. \tag{3.9}$$

We solve Eq.(3.9) for α_1, Eq.(3.8) for α_0 and Eq.(3.7) for q'. We obtain (choosing one solution of Eq.(3.9)

$$\alpha_1 = p\sqrt{\frac{-6e}{c}} \tag{3.10}$$

$$\alpha_0 = -\frac{b}{2c} + \frac{1}{2}p\lambda\sqrt{\frac{-6e}{c}} \tag{3.11}$$

$$q' = \frac{p}{4c}\left(-4ac + b^2 + 2cep^2\lambda^2 - 8ep^2\mu c\right) \qquad (3.12)$$

Now we substitute Eq.(3.9), Eq.(3.10), Eq.(3.11) into Eq.(3.6) and obtain $\alpha_1' = 0$ implies that

$$e(t) = r\,c(t) \qquad (3.13)$$

Where r is constant.

we substitute Eq.(3.9), Eq.(3.10), Eq.(3.11) into Eq.(3.5) and obtain $\alpha_0' = 0$ implies that

$$b(t) = s\,c(t) \qquad (3.14)$$

Where S is constant.

Therefore, the solution of the Eq.(3.5), Eq.(3.6), Eq.(3.7), Eq.(3.8) and Eq.(3.9) is as follows. We must assume Eq.(3.13) and Eq.(3.14) otherwise there is no solution. Then $q(t)$ is obtained from Eq.(3.12) by

$$q'(t) = \frac{p}{4}\left(-4a(t) + s^2c(t) + 2rp^2\lambda^2c(t) - 8rp^2\mu c(t)\right) \qquad (3.15)$$

Moreover, α_0 and α_1 are constant functions

$$\alpha_0(t) = \frac{1}{2}\left(-s + \sqrt{-6rp}\,\lambda\right) \qquad (3.16)$$

and

$$\alpha_1(t) = \sqrt{-6rp}. \qquad (3.17)$$

As an example, take

$$p = 1, r = -1, s = 1, \lambda = 0, \mu = -1$$

and

$$c(t) = t, \qquad a(t) = t^2, \qquad G(\xi) = \cosh\xi.$$

Then

$$\alpha_0 = -\frac{1}{2}, \quad \alpha_1 = \sqrt{6}, \quad q(t) = -\frac{1}{3}t^3 - \frac{7}{8}t^2.$$

We obtain that

$$u(t,x) = -\frac{1}{2} + \sqrt{6}\tanh\left(x - \frac{1}{3}t^3 - \frac{7}{8}t^2\right), \qquad (3.18)$$

with

$$\xi = x - \frac{1}{3}t^3 - \frac{7}{8}t^2$$

is a solution of equation Eq.(3.1). One can check with the computer that u given by Eq.(3.18) is really a solution of Eq.(3.1).

IV. CONCLUSION

This study shows that the generalized $\left(\frac{G'}{G}\right)$-expansion method is quite efficient and practically will suited for use in finding exact solutions for the problem considered here. New and more general excat solutions for any arbitrary functions $a(t), b(t), c(t)$ and $e(t)$ are obtained, there is no any method before, gave any exact solution for this equation. Also we construct an innovative explicit traveling wave solutions involving parameters of the generalized combined KdV and mKdV equation.

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