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On The Analysis of Aligned MHD Plane Flow in Porous Media in Presence of Magnetic Field

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Abstract
An attempt is made to study the steady MHD plane aligned flow through porous media in the presence of a magnetic field. An alternative approach to the Riabouchinsky method is developed for this flow problem. The proposed method will reduce the number of arbitrary constants arising when using the Riabouchinsky method. Consequently, many of the restrictive assumptions used in assigning values to the arbitrary constants are no longer needed. The analytical solution of the compatibility equations are determined leading to the solution of the velocity components and the pressure distribution for this problem.

Keywords: Aligned flow, porous medium, Riabouchinsky flow, vorticity-stream function.

1. Introduction
In recent years, the study of flow through porous media has generated considerable interest because of its numerous applications in the basic and applied sciences, including the study of groundwater flow movements in the porous earth layer, irrigation problems, the prediction of oil reservoir behavior, and the biophysical sciences, where the human lungs, for example are modeled as porous layers [5,8].
One of the particular importance of this work is the flow through porous media in the presence of a magnetic field. This type of flow may find some industrial applications in the design of filtration systems, in addition to the study of lubrication mechanism is enhanced through the introduction of porous lining into the mechanism and the imposition of a magnetic field in a transverse direction to the flow. Other applications to this type of flow include the occurrence of this type of flow in nature. Typically, the magnetic field is available everywhere on earth, hence, the study of any natural flow phenomena mandates taking into account the magnetic effects in the flow equations. The interest in this field dates as far back as 1865 by Darcy and currently one finds various models governing different types of fluid flow in various porous structures. The main type of single-phase flow models have been developed and reviewed by [6]. A vast amounts of research has been carried out on the motion of electrically conducting fluids moving in a magnetic field. Mathematical complexity of the phenomenon induced many researches to adopt a rather useful alternative technique of investigating special classes of flows such as aligned or parallel flows, crossed or orthogonal flows, (for more details see for example [1-4]). They studied finitely conducting orthogonal MHD plane flows. In which they discussed that the velocity and magnetic field vectors are mutually orthogonal everywhere in the flow region. In this paper, we consider aligned fluid flow through porous media in the presence of a magnetic field. The aim is to analyze the nonlinear model in an attempt to find possible solutions corresponding to a particular form of the stream function. The choice of the stream function in this work is one that is linear with respect to one of the independent variables. This type of flow is referred to as the Riabouchinsky flow. In this case, the two dimensional flow equations, written as a fourth order partial differential equations in terms of stream function, may be replaced by fourth order ordinary differential equations in two unknown functions of a single variable. Solutions to the coupled set are then obtained based on the knowledge of particular integrals of one of the equations. A different type of flow may then be studied with the knowledge of one of the functions. Riabouchinsky [11] assumed one of the functions to be zero and studied the resulting flow which represents a plane flow in which the flow is separated in the two symmetrical regions by a vertical or a horizontal plane. In addition to the study of Navier-Stokes flows and their applications [12], Riabouchinsky flows have also received considerable attention in the study of non-Newtonian flows [7] and in magnetohydrodynamics [9]. The approach used to solve the resulting coupled set of order ordinary differential equations suffers from some limitations, among which is its dependence on the knowledge of the particular solutions involve a number of arbitrary constants, the determination of which usually involves making many restrictive assumptions on the flow. A major disadvantage of the previous traditional approach is its inapplicability to the analysis of the Darcy-Lapwood-Brinkman (DLB) model. In this work, a modest modifications of the previous approach is employed and a methodology that is capable of handling a plane aligned flow problem is developed. The developed methodology overcomes some of the disadvantages of the traditional approach used for the Navier-Stokes equations and the Brinkman-type flow problems in porous media [10].

2. Aligned MHD Plane Flow
2.1 Basic Equations
The steady plane flow of an incompressible electrically conducting viscous fluid of infinite electrically conductivity in the presence of a magnetic field is governed by the following system of partial differential equations:

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$$div\ \vec{V}=0 \tag{2.1}$$

$$\rho(\vec{V}\cdot grad\ \vec{V})=-grad\ p+\mu_1\nabla^2\vec{V}-\frac{\mu_2}{k_1}\vec{V}+k_2(curl\ \vec{H}\times\vec{H}) \tag{2.2}$$

$$curl\ (\vec{V}\times\vec{H})-\frac{1}{k_2\sigma}curl\ curl\ \vec{H})=0 \tag{2.3}$$

$$div\ \vec{H}=0. \tag{2.4}$$

Where $\vec{V}$ is the velocity vector field, $\vec{H}$ is the magnetic vector field, p is the pressure, σ is the electrical conductivity, k_1 is the medium permeability to the fluid, k_2 is the magnetic permeability, μ_1 is the fluid viscosity, μ_2 is the viscosity of fluid in the porous medium and ρ is the fluid density. It is required to solve this system for the unknowns $\vec{V}$, $\vec{H}$ and p.

2.2 Simplifying the governing equations

We consider a two-dimensional aligned flow x and y with the velocity field $\vec{V}=(u,v,0)$, a magnetic field $\vec{H}=(0,0,H)$ and $\frac{\partial}{\partial z}=0$.

From the definition of aligned flows $\vec{H}$ and $\vec{V}$ are related by

$$\vec{H}(x,y)=F(x,y)\vec{V} \tag{2.5}$$

Where

$$(\vec{V}\cdot\nabla)F=0. \tag{2.6}$$

Setting equation (2.5) into equation (2.2) we get

$$\begin{aligned}(\nabla\times\vec{H})\times\vec{H}=[&-F^2v(v_x-v_y)-v^2FF_x+uv\ FF_y,\\ &F^2u(v_x-v_y)-u^2FF_y+uv\ FF_x,0]\end{aligned} \tag{2.7}$$

Substituting equation (2.5) into equation (2.3) we obtain

$$\nabla\times(\vec{V}\times\vec{H})=0. \tag{2.8}$$

If $\frac{1}{k_2\sigma}\neq 0$, then

$$\nabla\times(\nabla\times H)=\nabla^2H=[\nabla^2(Fu),\nabla^2(Fv),0)=0$$

or

$$\nabla^2(Fu)=\nabla^2(Fv)=0. \tag{2.9}$$

Inserting equation (2.5) into equation (2.4) we have

$$uF_x+vF_y=0. \tag{2.10}$$

Using the results (2.7)-(2.9) into equation (2.2) yields

$$\rho\{[\tfrac{1}{2}q^2]_x - v(v_x - u_y)\} = -P_x + \mu_1\nabla^2 u - \frac{\mu_2}{k_1}u + k_2[-F^2 v(v_x - u_y) - v^2 FF_x + uvFF_y] \tag{2.11}$$

$$\rho\{[\tfrac{1}{2}q^2]_y + u(v_x - u_y)\} = -p_y + \mu_1\nabla^2 u - \frac{\mu_2}{k_1}v + k_2[F^2 u(v_x - u_y) - u^2 FF_y + uv\, FF_x] \tag{2.12}$$

where $q^2 = u^2 + v^2$ is the square of the velocity field. Substituting equation (2.10) into equation (2.11) and (2.12) gives

$$\rho\{[\tfrac{1}{2}q^2]_x - v(v_x - u_y)\} = -p_x + \mu_1\nabla^2 u - \frac{\mu_2}{k_1}u + k_2[-F^2 v(v_x - u_y) - (u^2 + v^2)FF_x] \tag{2.13}$$

$$\rho\{[\tfrac{1}{2}q^2]_y + u(v_x - u_y)\} = -p_y + \mu_1\nabla^2 \mathrm{v} - \frac{\mu_2}{k_1}v + k_2[F^2 u(v_x - u_y) - (v^2 + u^2)FF_y]. \tag{2.14}$$

Then equation (2.1) takes the form

$$u_x + v_y = 0, \tag{2.15}$$

Thus the system of equations (2.1) to (2.4) are the flow equations governing the motion of a steady plane aligned flow of a viscous incompressible fluid of finite electrical conductivity. Equations (2.9), (2.13), (2.14) and (2.15) are the flow equations with the presence of a magnetic field.

In case of an infinitely electrical conductivity flow $(\frac{1}{k_2\sigma} \to 0)$ the diffusion equation (2.9) is identically satisfied. However, for finitely electrical conductivity flows, the functions F, u and v must satisfy the additional condition (2.6).

2.3 Vorticity-stream function form of equations

To derive the compatibility or integrability equations for aligned flows by employing the flow equations given in section 2.2 , where the stream function is linear with respect to x and y , we introduce the vorticity function $\omega(x,y)$ and the pressure function $h(x,y)$ defined respectively by

$$\omega(x,y) = v_x - u_y \tag{2.16}$$

$$h(x,y) = \mathrm{p} + \tfrac{1}{2}\rho q^2, \tag{2.17}$$

Then equations (2.13) and (2.14) take the following forms respectively:

$$h_x - \rho\, \mathrm{v}\,\omega = \mu_1\nabla^2 u - \frac{\mu_2}{k_1}u + k_2[-F^2 v\,\omega - (u^2 + v^2)FF_x] \tag{2.18}$$

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$$h_y - \rho u \omega = \mu_1 \nabla^2 v - \frac{\mu_2}{k_1} v + k_2[-F^2 u \omega - (u^2 + v^2) F F_y] \quad (2.19)$$

With the help of equations (2.15) and (2.16), equations (2.18) and (2.19) yield respectively

$$h_x - \rho v \omega = \mu_1 \omega_y - \frac{\mu_2}{k_1} u + k_2[-F^2 v \omega - (u^2 + v^2) F F_x] \quad (2.20)$$

$$h_y + \rho u \omega = \mu_1 \omega_x - \frac{\mu_2}{k_1} v + k_2[-F^2 u \omega - (u^2 + v^2) F F_y] \quad (2.21)$$

Now, letting $\psi(x, y)$ be the stream function defined in terms of the components as

$$\psi_y = u \text{ and } \psi_x = -v. \quad (2.22)$$

Then we can see clearly that the continuity equation is identically satisfied since

$$u_x + v_y = \psi_{xy} - \psi_{yx} = 0$$

And the vorticity equation (2.16) becomes

$$\omega = -\nabla^2 \psi . \quad (2.23)$$

Equations (2.20) and (2.21) are then become in terms of the stream functions as follows:

$$h_x = \rho \psi_x \nabla^2 \psi - \mu_1 (\nabla^2 \psi)_y - \frac{\mu_2}{k_1} \psi_y - k_2[F^2 \psi_x \nabla^2 \psi - (\psi_y^2 + \psi_x^2) F F_x] \quad (2.24)$$

$$h_y = \rho \psi_y \nabla^2 \psi + \mu_1 (\nabla^2 \psi)_x - \frac{\mu_2}{k_1} \psi_x + k_2[-F^2 \psi_y \nabla^2 \psi - (\psi_y^2 + \psi_x^2) F F_y]. \quad 2.25)$$

A compatibility equation can be derived from equations (2.24) and (2.25) by using the integrability condition $h_{xy} = h_{yx}$.

Thus, if we differentiate equation (2.24) with respect to y , and equation (2.25) with respect to x and using the integrability condition we obtain the compatibility equation

$$\rho \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} - k_1 F^2 \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} + k_1 F \frac{\partial(|\nabla \psi|^2, F)}{\partial(x, y)} - \frac{\mu_2}{k_1} \nabla^2 \psi + \mu_1 \nabla^4 \psi = 0 \quad (2.26)$$

where $\nabla^2 = \partial_{xx} + \partial_{yy}$, $\nabla^4 = \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy}$.

3. Reduction of the governing partial differential equations to ordinary differential equations

Once equation (2.26) is solved for $\psi(x, y)$, the vorticity function can then be calculated from equation (2.23), the pressure function $h(x, y)$ can be obtained from equation (2.24) and (2.25) while $p(x, y)$ is obtained from equation (2.17) and the velocity components can be determined from equation (2.22). Assume that the stream function $\psi(x, y)$ is linear and has the general form

$$\psi(x, y) = y\, f(x) + g(x), \quad (3.1)$$

where f and g are four times differentiable arbitrary functions. Substituting equation (3.1) into equation (2.26) and equating the coefficients of similar powers of y , lead to the following coupled set of fourth ordinary differential equations

$$\mu f^{(1v)} - \frac{\mu_2}{\rho k_1} f'' + (\rho - k_1 F^2)[f' f'' - f f'''] = 0 \quad (3.2)$$

$$\mu g^{(1v)} - \frac{\mu_2}{\rho k_1} g'' + (\rho - k_1 F^2)[g' f'' - f g'''] = 0. \quad (3.3)$$

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For the particular case of aligned flows $k_1 \to \infty$, we get

$$\frac{\partial(\mathrm{F}, \psi)}{\partial(x, y)} = 0. \tag{3.4}$$

The solution of equation (3.4) is either

$$\mathrm{F}(x, y) = c \quad \text{or} \quad \mathrm{F}(x, y) = \mathrm{F}(\psi),$$

where c is arbitrary constant.

If $\mathrm{F}(x, y) = c$, then equations (2.24), (2.25), (3.2) and (3.3) take the following forms respectively:

$$h_x = \rho \psi_x \nabla^2 \psi - \mu_1 (\nabla^2 \psi)_y - k_2 [f^{\,2} \psi_x \nabla^2 \psi] \tag{3.5}$$

$$h_y = \rho \psi_y \nabla^2 \psi + \mu_1 (\nabla^2 \psi)_x - k_2 [f^{\,2} \psi_y \nabla^2 \psi] \tag{3.6}$$

$$\mu f^{(1\mathrm{v})} + (\rho - k_1 c^2)[f' f'' - f f'''] = 0 \tag{3.7}$$

$$\mu g^{(1\mathrm{v})} + (\rho - k_1 c^2)[g' f'' - f g'''] = 0\,. \tag{3.8}$$

4. An alternative approach to the Riabouchinsky method

Using alternative approach presented in [10], two choices for the function $g(x)$ can be considered.

Case 1. $g(x) = g_1(x) = \alpha x^3$, where α is constant, then we obtain the corresponding equations:

$$f(x) = \frac{6\mu_1}{(\rho - k_1 c^2)} x^{-1} \tag{4.1}$$

$$\psi(x, y) = \frac{-6\mu_1}{(\rho - k_1 c^2)} x^{-1} y + \alpha x^3 \tag{4.2}$$

$$\omega(x, y) = \frac{12\mu_1}{(\rho - k_1 c^2)} x^{-3} y - 6\alpha x \tag{4.3}$$

$$u(x, y) = \frac{-6\mu_1}{(\rho - k_1 c^2)} x^{-1} \tag{4.4}$$

$$v(x, y) = \frac{-6\mu_1}{(\rho - k_1 c^2)} x^{-2} y - 3\alpha x^2 \tag{4.5}$$

$$\begin{aligned} h(x, y) = {} & \frac{-6\mu_1}{(\rho - k_1 c^2)} x^{-2} + \frac{18\mu_1^2}{(\rho - k_1 c^2)} x^{-4} y^2 + \frac{9}{2}(\rho - k_1 c^2)\, \alpha^2 x^4 \\ & + \frac{3\mu_1}{(\rho - k_1 c^2)} x^{-2} y^2 + 3\alpha x^2 y - 36\mu_1 \alpha y + \frac{36\mu_1^2}{(\rho - k_1 c^2)} x^{-4} \end{aligned} \tag{4.6}$$

$$\mathrm{p}(x, y) = \frac{-6\mu_1}{(\rho - k_1 c^2)} x^{-2} + \frac{18\mu_1^2}{(\rho - k_1 c^2)} x^{-4} y^2 + \frac{9}{2}(\rho - k_1 c^2)\, \alpha^2 x^4$$

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$$+\frac{3\mu_1}{(\rho-k_1c^2)}x^{-2}y^2+3\alpha x^2y-36\mu_1\alpha y+\frac{36\mu_1^2}{(\rho-k_1c^2)}x^{-4}$$
$$-\frac{18\rho\mu_1^2}{(\rho-k_1c^2)}x^{-2}-\frac{18\rho\mu_1^2}{(\rho-k_1c^2)}x^{-4}y^2-\frac{18\rho\alpha\mu_1}{(\rho-k_1c^2)}y-\frac{9}{2}\rho\alpha^2x^4 \;. \tag{4.7}$$

Case 2. $g(x)=g_2(x)=\alpha e^{\beta x}$, where α and β are constants, then similar to the previous procedures we obtain the corresponding equations

$$f(x)=c_1e^{\beta x}+\frac{\mu_1\beta}{(\rho-k_1c^2)}, \tag{4.8}$$

Where c_1 is arbitrary constant that can be determined by imposing one condition on the stream function ψ or by assuming various values to produce different flow patterns:

$$\psi(x,y)=c_1e^{\beta x}+\frac{\mu_1\beta}{(\rho-k_1c^2)}y+\alpha e^{\beta x} \tag{4.9}$$

$$\omega(x,y)=-\beta^2(\alpha+c_1y)e^{\beta x} \tag{4.10}$$

$$u(x,y)=\psi_y(x,y)=c_1e^{\beta x}+\frac{\mu_1\beta}{\rho-k_1c^2} \tag{4.11}$$

$$v(x,y)=-\psi_x(x,y)=-\beta(\alpha+c_1y)e^{\beta x} \tag{4.12}$$

$$h(x,y)=-c_1\mu_1\beta e^{\beta x}-\tfrac{1}{2}(\rho-k_1c^2)\alpha^2\beta^2e^{2\beta x}y$$
$$+(\rho-k_1c^2)c_1\beta^2e^{2\beta x}y+(\rho-k_1c^2)c_1^2\beta^2e^{2\beta x}y \tag{4.13}$$

$$\mathrm{p}(x,y)=-c_1\mu_1\beta e^{\beta x}-\tfrac{1}{2}(\rho-k_1c^2)\alpha^2\beta^2e^{2\beta x}+(\rho-k_1c^2)c_1\alpha\beta^2e^{2\beta x}y$$
$$+(\rho-k_1c^2)c_1^2\beta^2e^{2\beta x}y+\frac{2c_1\mu_1\beta}{(\rho-k_1c^2)}e^{\beta x}+\frac{\mu_1^2\beta^2}{(\rho-k_1c^2)}+c_1^2\beta^2e^{2\beta x}y$$
$$+2c_1\beta^2\alpha e^{2\beta x}y+\alpha^2\beta^2e^{2\beta x}. \tag{4.14}$$

5. Conclusion

In the current work we have presented analytical solutions for aligned MHD plane flow in porous media in the presence of a magnetic field. We have addressed some modifications to the solution methodology of the Navier-Stokes equations when the stream function is linear of the form $\psi(x,y)=y\,f(x)+g(x)$. This approach has the following advantages:

- The number of arbitrary constants arising when using the proposed method is considerably less than that for the Riabouchinsky method. Hence, many of the restrictive assumptions used in assigning values to the arbitrary constants are no longer needed.
- This new approach extends the ability to handle other flow models through porous medium in addition to the possibility of generating more solutions due to the method flexibility in choosing $g(x)$. It should be observed that this approach is also valid when the stream function is of the form $\psi(x,y)=x\,f(y)+g(y)$.

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